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PAIR CREATION IN THE EXTERNAL FIELD PROBLEM

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled PAIR CREATION IN THE EXTERNAL FIELD PROBLEM submitted by Bonnie Jean Edwards in partial fulfillment of the requirements for the degree of Master of Science in Theoretical Physics.

ABSTRACT

The phenomenon of pair creation of spin one half particles in an external electromagnetic field is examined. In particular, the concept of conservation of energy and momentum during pair production is studied.

The use of Bogoliubov transformations in scattering and creation calculations is reviewed. This formalism provides an expression for the pair creation amplitude and indicates that the c-number time translation operator for the problem is the quantity of chief interest.

A perturbation expression is found for the time translation operator, the Bogoliubov matrices and the pair creation amplitude. Both energy and momentum are conserved (to first order in the coupling constant e) during pair creation.

The time translation and symmetry properties of the time translation operator are investigated. The facts so acquired are not sufficient to discuss the energy and momentum conservation during pair creation using the exact solution.

Lastly, a few comments are made on the adiabatic theorem and what its validity would require of pair creation events in an external field.

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Notation

Many different conventions are seen in the literature of relativistic quantum field theory. To avoid later confusion, the conventions used in this work are described in this section.

Natural units where $\hbar = 1$ and $c = 1$ are used throughout.

A four vector has the following components:

$$x = (x^o, x^1, x^2, x^3) = (t, \underset{\sim}{x}) \ .$$

In discussing components of a four vector or tensor, Greek indices may take values from 0 to 3, while Latin indices may only take values from 1 to 3.

The metric tensor $g_{\mu\nu}$ is defined:

$$g_{oo} = 1, \quad g_{ii} = -1, \quad g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \ .$$

Using this metric a scalar product of two four vectors is defined:

$$x.y = g_{\mu\nu} x^\mu y^\nu = x^o y^o - \underset{\sim}{x}.\underset{\sim}{y} \ .$$

The Dirac matrices γ^μ are 4×4 constant matrices such that:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \ .$$

Other matrices of interest are:

$$\underset{\sim}{\sigma} = \frac{1}{2} \underset{\sim}{\gamma} \times \underset{\sim}{\gamma} \quad ,$$

$$\gamma_5 = \gamma_o\gamma_1\gamma_2\gamma_3 \cdot$$

The sixteen matrices I, $\underset{\sim}{\gamma}$, γ_o, $\gamma_o\underset{\sim}{\gamma}$, $\gamma_o\gamma_5$, $\underset{\sim}{\sigma}$, γ_5, $\gamma_o\underset{\sim}{\sigma}$ span the set of 4×4 complex matrices.

Whenever an explicit representation of the Dirac matrices is required, the following will be used:

$$I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} , \qquad \underset{\sim}{\gamma} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{pmatrix} ,$$

$$\gamma_o = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \qquad \gamma_o\gamma = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma} & 0 \end{pmatrix} ,$$

$$\gamma_o\gamma_5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} , \qquad \underset{\sim}{\sigma} = \begin{pmatrix} -i\underset{\sim}{\sigma} & 0 \\ 0 & -i\underset{\sim}{\sigma} \end{pmatrix} ,$$

$$\gamma_5 = \begin{pmatrix} 0 & -iI \\ -iI & 0 \end{pmatrix} , \qquad \gamma_o\underset{\sim}{\sigma} = \begin{pmatrix} -i\underset{\sim}{\sigma} & 0 \\ 0 & i\underset{\sim}{\sigma} \end{pmatrix} ,$$

where $\underset{\sim}{\sigma}$ are the 2×2 Pauli spin matrices.

CHAPTER I

THE EXTERNAL FIELD PROBLEM

1) Statement of the Problem

The system to be considered consists of massive relativistic particles moving under the influence of an external electromagnetic field but exerting no force on one another. In particular, it is the case of spin 1/2 fermions that will be investigated. For simplicity the particle will be called an electron and the anti-particle a positron. The mass of the electron is m.

The electromagnetic field is described by a potential $A(x)$, the quantized electron-positron field by $\psi(x)$ and the resulting Dirac equation describing the behaviour of $\psi(x)$ is:

$$(-i\gamma.\partial + m)\psi(x) = e\gamma.A(x)\psi(x) \quad . \tag{1.1}$$

Notice that $A(x)$ is specified, not calculated from the field equations - this is the meaning of "external". The mathematical nature of ψ and A will be commented on shortly.

Lacking interaction of the particles with themselves and reaction of the electromagnetic field to the particle field, this problem is sufficiently unrealistic to make it of little use as a model for elementary

particles. It does however have other redeeming properties. From a mathematical point of view, this problem is much simpler than a fully interacting field theory. In fact many physically interesting quantities such as creation and scattering amplitudes may actually be calculated. Many of the physical concepts desirable in a fully interacting theory have been incorporated here and may be examined carefully at this stage of sophistication - Lorentz invariance[1)] and spectral conditions are among these. If this problem is not physically reasonable, there is little chance for its more complicated version.

The particular phenomenon of interest in this thesis is creation. Is it possible to create an electron, a positron, a pair or even many pairs in an external field? As will be seen, no creation occurs in a time independent potential; in a time dependent potential it may possibly. In a physically sound theory it takes energy to create a particle of mass m. Is a threshold somehow built into the pair creation amplitude so that if the external field doesn't have enough energy (at least 2m), no pairs are created? In what sense does the external field have this energy? Similarly the momentum imparted to the pair must in some sense have come from the external field. What is the proper sense? Examination of the Fourier components of the potential A(x) seems indicated. Derivation of creation amplitudes and study of the

transformations involved in calculating these amplitudes will be carried out. The relation between creation and Fourier components of the field will be examined where possible.

The mathematical formalism for the time development of the particle field and the solution for creation amplitudes were the main body of Gilles Labonté's [1973] doctorate research and Chapters II and III of this thesis are to a large extent a review of his work. When of particular value, some of his proofs will be given; when little physical insight is gained through a careful mathematical proof, results alone will be adopted. In the later part of the thesis, more details will be presented.

The remainder of Chapter I involves setting the mathematical scene for the problem. The Hilbert space involved will be described. The nature of ψ and A will be made more clear.

Chapter II will discuss a general formulation for describing time development of the particle field in both time independent and time dependent external fields. The Bogoliubov transformations here defined will be an essential tool in calculating creation amplitudes.

Chapter III actually gives the formula for creation amplitudes. In the case of a weak external field where only pair creation and no isolated creation occurs,

a simplification will be found: it is only necessary to know the creation amplitude for one pair, to know all possible n-pair creation amplitudes. The one-pair creation problem will be set up and its examination begun.

In Chapter IV, the search for u, the c-number time translation operator, begins. An integral equation for u will be derived. The one-pair creation probability is calculated to first order in coupling constant using a perturbation expansion for u. Fourier components of the external field will be seen to appear as desired.

Chapter V is concerned solely with finding the time translation operator u. All of u's differential and symmetry properties are exploited.

In Chapter VI, a further attempt is made to calculate the pair creation amplitude explicitly.

Chapter VII will discuss the Adiabatic Theorem in the context of the external field problem. Time dependence is necessary for creation - will the time dependence introduced through adiabatic switching cause problems?

2) The Quantized Field Operator

In a quantum field theory, the field ψ is not an operator but an operator-valued distribution. In order to be interpreted as an operator the field ψ must be "smeared" with a sufficiently smooth and sufficiently quickly decreasing function h(x):

$$\hat{\psi}(h) = \int d^4x \ \psi(x)h(x) \quad .$$

Field equations, such as the Dirac equation (1.1), must be interpreted in a distributional sense.

One of the beauties of the external field problem is that $\psi(\underset{\sim}{x},t)$ is only a distribution in the spacial variables $\underset{\sim}{x}$ (Labonté [1973], Capri [1969]). It is, therefore, possible to refer to the field at particular times, compare the field at different times and in general handle the time dependence of field more easily than if it were the distribution in both space and time found in fully interacting theories.

Throughout this work, smearing will not be explicitly carried out. G. Labonté [1973] has shown that the required smearing is easily accomplished but the increase in the algebra tends to obscure rather than clarify the physics. Without smearing, some wave functions will be normalizable only to δ-functions. For simplicity the language of operators and functions will be used rather

than that of operator-valued distributions and function-valued distributions.

$\psi(x)$ is a quantized Heisenberg field operator on a Fock Hilbert space $\mathcal{H}$ of states. It will be seen that, at any given time t, $\psi(\underset{\sim}{x},t)$ may be expressed in terms of creation and annihilation operators acting on this Fock Hilbert space:

$$\psi(\underset{\sim}{x},t) = \sum_{\beta} f^{t}_{+\beta}(\underset{\sim}{x}) b_{\beta}(t) + f^{t}_{-\beta}(\underset{\sim}{x}) d^{\dagger}_{\beta}(t) \quad .$$

The q-number operators $b_{\beta}(t)$ and $d_{\beta}(t)$ are annihilation operators of the vacuum state $|0(t)\rangle$ of $\mathcal{H}$. The operators $b^{\dagger}_{\beta}(t)$ and $d^{\dagger}_{\beta}(t)$ acting on the vacuum give the one-electron and one-positron states respectively. It is conceivable that the vacuum and these annihilation operators will change with time as the external field fluctuates - it is for this reason that the time dependence is indicated.

The $f^{t}_{\varepsilon\beta}(\underset{\sim}{x})$ are elements of $[\mathcal{L}^2(R^3)]^4$ and are c-number wave functions of an appropriate Dirac equation at time t. The question of which Dirac equation is appropriate is examined in Chapter II. The c-number wave functions $f^{t}_{\varepsilon\beta}$ are orthonormal in their indices ε,β and form a basis for $[\mathcal{L}^2(R^3)]^4$.

The indices ε,β require further explanation. If ε is $+(-)$, the solution is that of an electron (positron) and corresponds to a positive (negative) energy solution

of the appropriate Dirac equation. The β index contains all other quantum numbers such as spin and momentum. Bound states will have discrete values of β; for scattering states, β will range over a continuum. A sum or integral over β will mean summation over discrete values of β and integration over continuous values.

Since $\psi(\underset{\sim}{x},t)$ may be examined for each t, a particle description of the system is possible at all times. At each time the vacuum state and particle states are defined and it will be shown that field equation (1.1) will help to relate these definitions at one time to those at another.

3) The c- and q-number Operators

Two types of operators occur in the discussion of the external field problem. The c-number operators act on elements of $[\mathcal{L}^2(R^3)]^4$ and will be denoted by lower case symbols (h, u, and so on). The q-number operators which are constructed out of the creation and annihilation operators act on the Fock Hilbert space $\mathcal{H}$ of particle states and will be denoted by upper case symbols (H, U, Q, N and so on). Often a calculation may be accomplished with either a c-number or a q-number operator but for explicit work the c-number operator is usually more tractable.

In the external field problem, the desired q-number operators are well-defined at all times;

$$H(t) = \sum_{\beta} \omega_{\beta}(t)(b_{\beta}^{\dagger}(t)b_{\beta}(t) + d_{\beta}^{\dagger}(t)d_{\beta}(t))$$

$$Q(t) = e\sum (b_{\beta}^{\dagger}(t)b_{\beta}(t) - d_{\beta}^{\dagger}(t)d_{\beta}(t))$$

$$N(t) = \sum_{\beta} (b_{\beta}^{\dagger}(t)b_{\beta}(t) + d_{\beta}^{\dagger}(t)d_{\beta}(t)) \quad .$$

This fact simply reflects the feasibility of a particle description at all times. That these operators are seldom mentioned in this work is just a matter of convenience of computation.

4) The External Field A(x)

The necessary mathematical criteria for A(x) are not certain. It is sufficient for A(x) to be a smooth function of x of fast decrease; that is, it may be an infinitely differentiable function falling to zero at infinity in all directions of space-time faster than any inverse polynomial. G. Labonté [1973] has successfully worked examples where A(x) is not of this form. For instance, if the external field is purely electric with continuous,step function or even δ-function time dependence very bad spacial behaviour may be tolerated. It is through the inequalities that the Bogoliubov transformations must obey, that restrictions on A(x) are imposed.

In this thesis the external field will usually have continuous (or at worst step function) time dependence and it is assumed that only sufficiently well behaved fields are treated. Since the class of such external fields is quite large, this imposes little restriction on the possible fields to be examined.

CHAPTER II

TIME DEVELOPMENT OF PARTICLE FIELD

1) Time Independent Potentials

Consider the Dirac electron-positron field $\psi(x)$ interacting with a time independent external potential $A(\underset{\sim}{x})$:

$$(-i\gamma.\partial + m)\psi(\underset{\sim}{x},t) = e\gamma.A(\underset{\sim}{x})\psi(\underset{\sim}{x},t) \qquad (1.1)$$

i.e. $i\frac{\partial}{\partial t}\psi(\underset{\sim}{x},t) = h\psi(\underset{\sim}{x},t)$

where $h_o = \gamma^o(i\underset{\sim}{\gamma}.\underset{\sim}{\partial} + m)$,

and $h = h_o - e\gamma^o\gamma.A(\underset{\sim}{x})$.

The quantized field operator solution is:

$$\psi(\underset{\sim}{x},t) = e^{-iht}\,\psi(\underset{\sim}{x},0) \qquad (1.2)$$

where $\psi(\underset{\sim}{x},0) = \sum_\beta f_{+\beta}(\underset{\sim}{x})b_\beta + f_{-\beta}(\underset{\sim}{x})d^\dagger_\beta$. (1.3)

The $f_{\varepsilon\beta}$ are c-number solutions to:

$$hf_{\varepsilon\beta}(\underset{\sim}{x}) = \varepsilon\omega(\beta)f_{\varepsilon\beta}(\underset{\sim}{x}) \quad . \qquad (1.4)$$

The bound states $(\omega < m)$ must be calculated by hand but the scattering states $(\omega > m)$ may be obtained by application of the Möller operation Ω_+ to the corresponding free wave functions $f^o_{\varepsilon\beta}(\underset{\sim}{x})$. The free wave functions $f^o_{\varepsilon\beta}(\underset{\sim}{x})$ are:

$$f^o_{+\beta} = \frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} w^s(\underset{\sim}{p}) e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} ,$$

$$f^o_{-\beta} = \frac{1}{(2\pi)^{3/2}} \left(\frac{m}{\omega(\underset{\sim}{p})}\right)^{\frac{1}{2}} v^s(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} , \qquad (1.5)$$

where $\beta = (\underset{\sim}{p},s)$,

$$(\gamma^o\omega(\underset{\sim}{p}) - \underset{\sim}{\gamma}\cdot\underset{\sim}{p}-m)w^s(\underset{\sim}{p}) = 0 ,$$

$$(1.6)$$

$$(\gamma^o\omega(\underset{\sim}{p}) - \underset{\sim}{\gamma}\cdot\underset{\sim}{p}+m)v^s(\underset{\sim}{p}) = 0 ,$$

and where $\omega(\underset{\sim}{p}) = (\underset{\sim}{p}\cdot\underset{\sim}{p} + m^2)^{\frac{1}{2}}$.

The full wave functions are then:

$$f_{\varepsilon\beta}(\underset{\sim}{x}) = \Omega_+ f^o_{\varepsilon\beta}(\underset{\sim}{x}) \quad . \qquad (1.7)$$

Using (1.4) in (1.2):

$$\psi(\underset{\sim}{x},t) = \sum_\beta e^{-i\omega(\beta)t} f_{+\beta}(\underset{\sim}{x}) b_\beta + e^{i\omega(\beta)t} f_{-\beta}(\underset{\sim}{x}) d^\dagger_\beta .$$

The creation and annihilation operators satisfy the standard anticommutation relations with all anti-commutators but the following vanishing:

$$\{b_\beta, b^\dagger_{\beta'}\} = \delta(\beta,\beta')$$

$$(1.8)$$

$$\{d_\beta, d^\dagger_{\beta'}\} = \delta(\beta,\beta') \quad .$$

The vacuum state $|0>$ is such that

$$b_\beta|0> = d_\beta|0> = 0 \quad . \qquad (1.9)$$

and $<0|0> = 1$.

The action of $b_\beta^\dagger (d_\beta^\dagger)$ on the vacuum is to create an electron (positron) of type β.

Description of Scattering

A solution of the free field problem ($A \equiv 0$) is obtained following the same method as used above for the external field:

$$\psi_o(\underset{\sim}{x},t) = e^{-ih_o t} \psi_o(\underset{\sim}{x},0) \ ,$$

$$\psi_o(\underset{\sim}{x},0) = \sum_\beta b_\beta^o f_{+\beta}^o(\underset{\sim}{x}) + d_\beta^{o\dagger} f_{-\beta}^o(\underset{\sim}{x}) \tag{1.10}$$

where

$$b_\beta^o = \int d^3x \ (e^{-ih_o t} f_{+\beta}^o(\underset{\sim}{x}))^* \psi_o(\underset{\sim}{x},t) \tag{1.11}$$

$$d_\beta^{o\dagger} = \int d^3x \ (e^{-ih_o t} f_{-\beta}^o(\underset{\sim}{x}))^* \psi_o(\underset{\sim}{x},t) \tag{1.12}$$

and where b_β^o and d_β^o are defined on the Fock Hilbert space with vacuum state $|0>_o$.

Returning to the problem with external field $A(\underset{\sim}{x})$, an analysis of scattering is sought. For a discussion of scattering to be meaningful, $\psi(\underset{\sim}{x},t)$ must describe an asymptotically free field as $t \to \pm\infty$. An analogy with (1.11) and (1.12) suggests that the creation/annihilation operators defined by the following limits must exist:

$$\lim_{t\to\pm\infty} \int d^3x \ (e^{-ih_o t} f_{\varepsilon\beta}^o(\underset{\sim}{x}))^* \psi(\underset{\sim}{x},t) \ . \tag{1.13}$$

The limit $t \to -\infty$ gives b and $d^\dagger$ used to define incoming

particles and the limit $t \to +\infty$ gives b and $d^\dagger$ used to define outgoing particles. Care must be exercised in the calculation of (1.13) because the strong operator limit is desired. By (1.2):

$$\int d^3x\,(g(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},t) = \int d^3x\,(e^{iht}g(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) \qquad \forall\, g \in (\mathcal{L}^2)^4$$

so that

$$\left(\int d^3x\,(e^{-ih_ot} f(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},t) - \int d^3x\,(\Omega_\pm f(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0)\right)|\phi>$$

$$= \left(\int d^3x\,[(e^{iht}e^{-ih_ot} - \Omega_\pm) f(\underset{\sim}{x})]^*\psi(\underset{\sim}{x},0)\right)|\phi> \qquad \forall\, |\phi> \in \mathcal{H} \quad . \tag{1.14}$$

Now (1.8) gives:

$$\{\psi_\mu(\underset{\sim}{x},0), \psi^\dagger_\nu(\underset{\sim}{x}',0)\} = \delta_{\mu\nu}\delta(\underset{\sim}{x}-\underset{\sim}{x}') \quad ,$$

which in turn gives:

$$||\left(\int d^3x\,(g(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0)\right)|\phi>||^2 \leq (g,g)<\phi|\phi> \qquad \forall\, g \in (\mathcal{L}^2)^4,\ |\phi> \in \mathcal{H} \quad . \tag{1.15}$$

Combining (1.14) and (1.15)

$$\lim_{t\to\mp\infty} ||\left(\int d^3x\,(e^{ih_ot} f(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},t) - \int d^3x\,(\Omega_\pm f(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0)\right)|\phi>||$$

$$\leq \lim_{t\to\mp\infty} ||e^{iht}e^{-ih_ot} f(\underset{\sim}{x}) - \Omega_\pm f(\underset{\sim}{x})||\,<\phi|\phi>^{\frac{1}{2}} = 0 \qquad \forall\, |\phi> \in \mathcal{H}$$

by definition of $\Omega_\pm$. So:

$$\lim_{t\to\mp\infty}\int d^3x\,(e^{-ih_o t} f^o_{\varepsilon\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},t) = \int d^3x\,(\Omega_\pm f^o_{\varepsilon\beta}(\underset{\sim}{x}))^*\psi(0,\underset{\sim}{x}) \quad . \tag{1.16}$$

The incoming operators are:

$$\begin{aligned} b^{in}_\beta &= \int d^3x\,(\Omega_+ f^o_{+\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) \\ &= \int d^3x\,(f_{+\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) = b_\beta \\ d^{in\dagger}_\beta &= \int d^3x\,(\Omega_+ f^o_{-\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) = d^\dagger_\beta \quad . \end{aligned} \tag{1.17}$$

The outgoing operators are:

$$\begin{aligned} b^{out}_\beta &= \int d^3x\,(\Omega_- f^o_{+\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) \\ &= \sum_{\beta'} b_{\beta'}(\Omega_- f^o_{+\beta},\Omega_+ f^o_{+\beta'}) + d^\dagger_{\beta'}(\Omega_- f^o_{+\beta},\Omega_+ f^o_{-\beta'}) \\ d^{out\dagger}_\beta &= \int d^3x\,(\Omega_- f^o_{-\beta}(\underset{\sim}{x}))^*\psi(\underset{\sim}{x},0) \\ &= \sum_{\beta'} b_{\beta'}(\Omega_- f^o_{-\beta},\Omega_+ f^o_{+\beta'}) + d^\dagger_{\beta'}(\Omega_- f^o_{-\beta},\Omega_+ f^o_{-\beta'}) \quad . \end{aligned} \tag{1.18}$$

The scattering matrix is defined by $S = (\Omega_-)^\dagger\Omega_+$ and satisfies $[S,h_o] = 0$ so that:

$$(f^o_{+\beta},Sf^o_{-\beta}) = (f^o_{-\beta},Sf^o_{+\beta}) = 0 \quad . \tag{1.19}$$

With this relation, (1.18) becomes:

$$\begin{aligned} b^{out}_\beta &= \sum_{\beta'} b^{in}_{\beta'}(f^o_{+\beta},Sf^o_{+\beta'}) \\ d^{out\dagger}_\beta &= \sum_{\beta'} d^{in\dagger}_{\beta}(f^o_{-\beta},Sf^o_{-\beta'}) \quad . \end{aligned} \tag{1.20}$$

A Bogoliubov transformation is a linear relation among creation and annihilation operators - in this work, Bogoliubov transformations relate those at one time to those at another time. It is seen that (1.20) is a simple example of a Bogoliubov transformation. The sets $\{b^{in},d^{in}\}$, $\{b,d\}$ are $\{b^{out},d^{out}\}$ all operate on $\mathcal{H}$ and all annihilate the same vacuum $|0\rangle$. Only one vacuum is needed for all times and (1.20) shows that the notions of electron and positron don't become intermingled with time. Notice that bound states seem to play no role in the scattering problem: only continuum wave functions have been needed to describe the system as $t \to \pm\infty$. This is the same as in ordinary c-number quantum mechanics.

Scattering Amplitudes

Consider a system in the Heisenberg state $|\phi\rangle$. To determine what electrons and positrons make up the system at $t \to -\infty$, $|\phi\rangle$ must be expanded in terms of "in" states:

$$|\phi\rangle = \sum_{nm} \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} \int d^3\underline{\alpha}_n \; d^3\underline{\beta}_m \; \psi^{in}_{nm}(\underline{\alpha}_n,\underline{\beta}_m) b^{in\dagger}_{\alpha_1}\cdots b^{in\dagger}_{\alpha_n} d^{in\dagger}_{\beta_1}\cdots d^{in\dagger}_{\beta_m}|0\rangle . \tag{1.21}$$

To determine what the system consists of as $t \to +\infty$, $|\phi\rangle$ must be expanded in "out" states:

$$|\phi\rangle = \sum_{nm} \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} \int d^3\underline{\alpha}'_n d^3\underline{\beta}'_m \psi^{out}_{nm}(\underline{\alpha}'_n,\underline{\beta}'_m) b^{out\dagger}_{\alpha'_1}\cdots b^{out\dagger}_{\alpha'_n} d^{out\dagger}_{\beta'_1}\cdots d^{out\dagger}_{\beta'_m}|0\rangle . \tag{1.22}$$

To describe a system in the state $|\phi\rangle$ at any time t, $|\phi\rangle$ must be expanded in terms of the particle states as defined at that time t.

Consider a system consisting initially of n electrons with quantum numbers $\underline{\alpha}_n$ and m positrons with quantum numbers $\underline{\beta}_m$:

$$|\phi\rangle = b_{\alpha_1}^{in\dagger} \ldots b_{\alpha_n}^{in\dagger} d_{\beta_1}^{in\dagger} \ldots d_{\beta_m}^{in\dagger} |0\rangle$$

$$= \left(\int d^3\alpha_1' b_{\alpha_1'}^{out\dagger} (S^\dagger)_{+\alpha_1,+\alpha_1'}\right) \ldots \left(\int d^3\beta_m' d_{\beta_m'}^{out\dagger} (S)_{-\beta_m,-\beta_m'}\right) |0\rangle$$

$$= \int d^3\underline{\alpha}_n' \; d^3\underline{\beta}_m' \; \psi(\underline{\alpha}_n',\underline{\beta}_m') \; b_{\alpha_1'}^{out\dagger} \ldots . d_{\beta_m'}^{out\dagger} |0\rangle \qquad (1.23)$$

where

$$(S^+)_{+\alpha,+\alpha'} = (f^o_{+\alpha}, S^\dagger f^o_{+\alpha'}) \; ,$$

$$(S)_{-\beta,-\beta'} = (f^o_{-\beta}, S f^o_{-\beta'}) \; ,$$

and

$$\psi(\underline{\alpha}_n',\underline{\beta}_m') = \prod_{i=1}^{n} (S^\dagger)_{+\alpha_i,+\alpha_i'} \prod_{j=1}^{m} (S)_{-\beta_j,-\beta_j'} \; . \qquad (1.24)$$

Between $t = \pm\infty$, the system of n electrons and m positrons is acted on by the external field. As $t \to +\infty$, the quantum numbers of these particles have changed according to the scattering amplitude ψ of (1.24), but the numbers of electrons and positrons remain constant. It is seen from (1.23) that while the electrons and positrons may suffer

change of spin components or momentum by the action of a time independent external field, no creation or annihilation occurs. That no creation can occur in a time independent external field will be useful when attempting to discuss time dependent external fields.

2) Time Dependent Potentials

Solution of the Differential Equation

The quantized field operator solution of the following Dirac equation is sought

$$(-i\gamma\cdot\partial + m)\psi(\underset{\sim}{x},t) = e\gamma\cdot A(\underset{\sim}{x},t)\psi(\underset{\sim}{x},t)$$

i.e. $$i\frac{\partial}{\partial t}\psi(\underset{\sim}{x},t) = h(t)\psi(\underset{\sim}{x},t) \qquad (2.1)$$

where $h(t) = h_o - e\gamma^o\gamma\cdot A(\underset{\sim}{x},t)$.

G. Labonté [1973] has proved the existence of a c-number unitary time translation operator, $u(t,t')$, on $[\mathcal{L}^2(R^3)]^4$. This operator satisfies

$$i\frac{\partial}{\partial t}u(t,t') = h(t)u(t,t')$$

$$u(t,t) = 1$$

$$u(t_3,t_2)u(t_2,t_1) = u(t_3,t_1) \ . \qquad (2.2)$$

Notice in the case of time independent potentials,

$$u(t,t') = e^{-i(t-t')h} \ .$$

The properties listed in (2.2), lead to a solution of (2.1):

$$\psi(\underset{\sim}{x},t) = u(t,t')\psi(\underset{\sim}{x},t'). \tag{2.3}$$

Auxiliary Fields

The notions of vacuum state and particle state must be clarified in a situation of changing external field. If the system is in the vacuum state at one time, it is in the lowest energy state available at that time. When the external field fluctuates, this state may no longer be the lowest energy state: it is seen that the definition of the vacuum will change with time. Notice however, if a system is in a state (always Heisenberg states are used) which is the vacuum at one time it may not be the vacuum at a later time - creation may occur. How precisely does the meaning of "vacuum" and hence "particle" change with time in a time dependent external field?

To discuss the vacuum and particle states at time t_o, a new quantized field will be defined. This auxiliary field ψ_{t_o} satisfies:

$$(-i\gamma\cdot\partial + m)\psi_{t_o}(\underset{\sim}{x},t) = e\gamma\cdot A(\underset{\sim}{x},t_o)\psi_{t_o}(\underset{\sim}{x},t) \ , \tag{2.4}$$

and

$$\psi_{t_o}(\underset{\sim}{x},t_o) = \psi(\underset{\sim}{x},t_o) \ . \tag{2.5}$$

So ψ_{t_o} is the solution of a time independent problem and agrees with the actual field ψ at time t_o. ψ_{t_o} behaves in exactly the same way as ψ would have, if the external potential $A(\underset{\sim}{x},t)$ had been held constant at $A(\underset{\sim}{x},t_o)$ after time t_o. Because ψ_{t_o} is the quantized field of a time independent problem:

$$\psi_{t_o}(\underset{\sim}{x},t) = \sum_\beta f^{t_o}_{+\beta}(\underset{\sim}{x}) b^{t_o}_\beta e^{-i\omega(\beta)t} + f^{t_o}_{-\beta}(\underset{\sim}{x}) d^{t_o\dagger}_\beta e^{i\omega(\beta)t} \tag{2.6}$$

where

$$b^{t_o}_\beta e^{-i\omega(\beta)t} = \int d^3x (f^{t_o}_{+\beta}(\underset{\sim}{x}))^* \psi_{t_o}(\underset{\sim}{x},t) \ ,$$

$$d^{t_o\dagger}_\beta e^{i\omega(\beta)t} = \int d^3x (f^{t_o}_{-\beta}(\underset{\sim}{x}))^* \psi_{t_o}(\underset{\sim}{x},t) \ ,$$

and

$$b^{t_o}_\beta |0\rangle_{t_o} = d^{t_o}_\beta |0\rangle_{t_o} = 0 \ .$$

The following identification is made:

$$\begin{aligned} b_\beta(t_o) &= b^{t_o}_\beta e^{-i\omega(\beta)t_o} \\ d_\beta(t_o) &= d^{t_o}_\beta e^{-i\omega(\beta)t_o} \\ |0(t_o)\rangle &= |0\rangle_{t_o} \quad . \end{aligned} \tag{2.7}$$

Because of the initial condition (2.5), the q-number energy, charge and number operators of the actual and the auxiliary fields are the same at time t_o. The Heisenberg equations for ψ and ψ_{t_o} are also identical at time t_o. It is, therefore, meaningful to adopt (2.7) as the definition for the vacuum and particle annihilation operators of the actual field ψ at time t_o.

The auxiliary field can easily be decomposed into positive and negative frequency parts since it is the solution for a time independent external field. It is not clear how to make a comparable decomposition for the actual field because the external field varies with time. In the time dependent problem, the vacuum keeps changing and the frequencies in ψ may change as well - the notion of positive and negative frequencies becomes obscured in a time dependent problem. It is for this reason that auxiliary fields are needed.

Bogoliubov Transformations

Consider the quantized particle field at two times t_1 and t_2. At each time there will be an appropriate Fock Hilbert space $\mathcal{H}_t$ with vacuum $|0(t)\rangle$, operators $b_\beta(t)$ and $d_\beta(t)$, and c-number wave functions $f^t_{\varepsilon\beta}(\underset{\sim}{x})$. Equation (2.3) says:

$$\psi(\underset{\sim}{x},t_2) = u(t_2,t_1)\psi(\underset{\sim}{x},t_1) .$$

Decomposing each ψ as in (2.6) and using (2.7):

$$b_\beta(t_2) = \sum_{\beta'} (f^{t_2}_{+\beta},u(t_2,t_1)f^{t_1}_{+\beta'})b_{\beta'}(t_1) + (f^{t_2}_{+\beta},u(t_2,t_1)f^{t_1}_{-\beta'})d^\dagger_{\beta'}(t_1),$$

$$d^\dagger_\beta(t_2) = \sum_{\beta'} (f^{t_2}_{-\beta},u(t_2,t_1)f^{t_1}_{+\beta'})b_{\beta'}(t_1) + (f^{t_2}_{-\beta},u(t_2,t_1)f^{t_1}_{-\beta'})d^\dagger_{\beta'}(t_1) . \tag{2.8}$$

The cross terms in (2.8) may or may not be zero. Time dependence of the external field has led to a more complicated Bogoliubov transformation than (1.20). For

(2.8) to be meaningful, $|0(t_2)>$ must lie in $\mathcal{H}_{t_1}$ and hence $\mathcal{H}_{t_2} \subseteq \mathcal{H}_{t_1}$. Interchanging the roles of t_1 and t_2, it is seen that $\mathcal{H}_{t_1} = \mathcal{H}_{t_2}$ is required. An equivalent criterion is that (2.8) be unitarily implementable.

If the external field admits asymptotically free states for large times, (2.8) will hold for $t_1 \to -\infty$ and $t_2 \to +\infty$. In these limits, the quantities labelled by t_1 become "in" quantities and those labelled by t_2 become "out" quantities. Scattering calculations are now possible.

Creation Problem

Consider a system in a Heisenberg state $|\phi>$ which is the vacuum state at time t_1: the physical appearance of the system at time t_1 is that it contains no electrons or positrons. What does the system look like at a later time t_2?

$$|\phi> = |0(t_1)>$$

$$= \sum_{nm} \frac{1}{\sqrt{n!m!}} \int d\underline{\alpha}_n d\underline{\beta}_m \psi^{nm}(\underline{\alpha}_n, \underline{\beta}_m) b^\dagger_{\alpha_1}(t_2) \ldots d^\dagger_{\beta m}(t_2)|0(t_2)> \quad (2.9) .$$

The system at time t_2 is a superposition of many particle states. The amplitude for creation of n electrons and m positrons with quantum numbers $\underline{\alpha}_n$ and $\underline{\beta}_m$ respectively is $\psi^{nm}(\underline{\alpha}_n, \underline{\beta}_m)$. It is knowledge of $\psi^{nm}(\underline{\alpha}_n, \underline{\beta}_m)$ that is required

in order to discuss creation. Chapter III will demonstrate how to obtain $\psi^{nm}(\underline{\alpha}_n, \underline{\beta}_m)$ once the Bogoliubov transformation (2.8) is known.

3) Example: The Scalar Field with a c-number Source

This example illustrates the use of the auxiliary field in obtaining the time development of a system. The desired relation between creation amplitudes and Fourier components of the external field also emerges.

Consider a neutral scalar field ϕ in the presence of an external c-number source ρ. This problem may be handled by essentially the same method used for fermions. The equation of motion is:

$$(\Box + m^2)\phi(\underset{\sim}{x},t) = \rho(\underset{\sim}{x},t) \tag{3.1}$$

where $\rho(\underset{\sim}{x},t) = 0 \quad \forall \quad t < t_s$ and $t > t_f$.

The incoming field ϕ_{in} may be decomposed into creation and annihilation operators:

$$\phi_{in}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(k)}} \left(a^{in}_{\underset{\sim}{k}}\, e^{-ik.x} + a^{in\dagger}_{\underset{\sim}{k}}\, e^{ik.x}\right) , \tag{3.2}$$

where $k_o = \omega(k) = \sqrt{\underset{\sim}{k}.\underset{\sim}{k} + m^2}$.

The operators a and $a^\dagger$ are the standard boson annihilation and creation operators.

The retarded Green's function Δ_r is defined:

$$\Delta_r(x) = \theta(x_o) \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega(k)} e^{i\underset{\sim}{k}.\underset{\sim}{x}} \sin(\omega(k)t) \quad . \tag{3.3}$$

A solution to (3.1) is:

$$\phi(x) = \phi_{in}(x) + \int d^4y \; \Delta_r(x-y)\rho(y) \quad . \tag{3.4}$$

The auxiliary field needed to describe particles at time t_o satisfies:

$$(\Box + m^2)\phi_{t_o}(\underset{\sim}{x},t) = \rho(\underset{\sim}{x},t_o)$$

$$\phi_{t_o}(\underset{\sim}{x},t_o) = \phi(\underset{\sim}{x},t_o)$$

$$\frac{\partial}{\partial t}\phi_{t_o}(\underset{\sim}{x},t)\Big|_{t=t_o} = \frac{\partial}{\partial t}\phi(\underset{\sim}{x},t)\Big|_{t=t_o} \quad . \tag{3.5}$$

To obtain $\phi_{t_o}(x)$ for times $t > t_o$, use (3.4) with ρ kept constant at $\rho(\underset{\sim}{x},t_o)$ after time t_o:

$$\phi_{t_o}(x) = \phi_{in}(x) + \int_{-\infty}^{t_o} dy_o \; d^3y \; \Delta_r(x-y)\rho(\underset{\sim}{y},y_o)$$

$$+ \int_{t_o}^{\infty} dy_o \; d^3y \; \Delta_r(x-y)\rho(\underset{\sim}{y},t_o)$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(k)}} \left(a_{\underset{\sim}{k}}(t_o)e^{-ik.x} + a^\dagger_{\underset{\sim}{k}}(t_o)e^{ik.x}\right)$$

$$+ \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\omega^2(k)} \tilde{\rho}(\underset{\sim}{k},t_o)e^{i\underset{\sim}{k}.\underset{\sim}{x}}, \tag{3.6}$$

where

$$a_{\underset{\sim}{k}}(t_o) = a^{in}_{\underset{\sim}{k}} + \frac{i}{\sqrt{2\omega(k)}} \int_{-\infty}^{t_o} dy_o e^{i\omega(k)y_o}\tilde{\rho}(\underset{\sim}{k},y_o) - \frac{e^{i\omega(k)t}}{\sqrt{2}\,\omega(k)^{3/2}}\tilde{\rho}(\underset{\sim}{k},t_o), \tag{3.7}$$

and

$$\tilde{\rho}(\underset{\sim}{k},t_o) = \frac{1}{(2\pi)^{3/2}} \int d^3x \; e^{i\underset{\sim}{k}.\underset{\sim}{x}} \rho(\underset{\sim}{k},t_o).$$

As in the fermion development, the operators defined using the auxiliary field at time t_o may be used for the actual field at time t_o. The vacuum for ϕ_{t_o}, $|0(t_o)\rangle$, is the physical vacuum for ϕ at time t_o; a particle of momentum k at time t_o is $a_k^\dagger(t_o)|0(t_o)\rangle$. The Fock Hilbert spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{t_o}$ coincide.

If the system is initially a vacuum, what does it look like at time t?

$$|0\rangle_{in} = \alpha^o(t)|0(t)\rangle + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int \frac{d^3k_1}{\sqrt{2\omega(k_1)}} \cdots \frac{d^3k_n}{\sqrt{2\omega(k_n)}} \cdot$$

$$\alpha^n(k_1,\ldots k_n,t)\, a^\dagger_{\underset{\sim}{k}_1}(t)\ldots a^\dagger_{\underset{\sim}{k}_n}(t)|0(t)\rangle . \qquad (3.8)$$

Now $a^{in}_{\underset{\sim}{k}}$ annihilates $|0\rangle_{in}$, and $a_{\underset{\sim}{k}}(t)$ annihilates $|0(t)\rangle$. Calculating $a^{in}_{\underset{\sim}{k}}|0\rangle_{in} = 0$ in detail using (3.8) and (3.7), it is found that

$$\alpha^n(\underset{\sim}{k}_1,\ldots \underset{\sim}{k}_n,t) = \frac{(-1)^n}{\sqrt{n!}} \left(\prod_{i=1}^{n} \sqrt{2\omega(k_i)}\; S(\underset{\sim}{k}_i,t)\right)\alpha_o(t), \qquad (3.9)$$

where

$$S(\underset{\sim}{k},t) = \frac{-i}{\sqrt{2\omega(k)}} \int_{-\infty}^{t} dy_o e^{i\omega(k)y_o} \tilde{\rho}(\underset{\sim}{k},y_o) + \frac{e^{i\omega(k)t}}{\sqrt{2}\;\omega(k)^{3/2}} \tilde{\rho}(k,t).$$

Examining the system as $t \to +\infty$:

$$S^{out}(\underset{\sim}{k}) = \frac{-i}{\sqrt{2\omega(k)}} \int_{-\infty}^{\infty} dy_o e^{i\omega y_o} \tilde{\rho}(\underset{\sim}{k},y_o) \propto \tilde{\tilde{\rho}}(\underset{\sim}{k},-\omega(k)), \qquad (3.10)$$

where

$$\tilde{\tilde{\rho}}(\underset{\sim}{k}\ ,\omega(k)) = \frac{1}{(2\pi)^2}\int d^4x\ e^{-ik.x}\ \rho(x)\ .$$

For a particle of energy $\omega(k)$ and momentum k to be created and contribute to any of the amplitudes α^n of (3.9), the external field ρ must have a non-vanishing Fourier component $\tilde{\tilde{\rho}}(\underset{\sim}{k},-\omega(k))$. This is the "conservation" of energy and momentum law that was anticipated.

CHAPTER III

CREATION AMPLITUDES

1) General Solution

Bogoliubov Transformation

In Chapter II, it was seen that the creation and annihilation operators defined at one time t_2 are related to those at another time t_1 by a Bogoliubov transformation. The general form for this transformation is:

$$b_\gamma(t_2) = \int d\alpha M_1(\gamma,\alpha) b_\alpha(t_1) + \int d\beta M_2(\gamma,\beta) d^\dagger_\beta(t_1) \ ,$$

$$d^\dagger_\lambda(t_2) = \int d\alpha M_3(\lambda,\alpha) b_\alpha(t_1) + \int d\beta M_4(\lambda,\beta) d^\dagger_\beta(t_1) \ . \qquad (1.1)$$

The inverse transformation is:

$$b_\alpha(t_1) = \int d\gamma M^*_1(\gamma,\alpha) b_\gamma(t_2) + \int d\lambda M^*_3(\lambda,\alpha) d^\dagger_\lambda(t_2) \ ,$$

$$d^\dagger_\beta(t_1) = \int d\gamma M^*_2(\gamma,\beta) b_\beta(t_2) + \int d\lambda M^*_4(\lambda,\beta) d^\dagger_\lambda(t_2) \ . \qquad (1.2)$$

The annihilation and creation operators at one time satisfy standard fermion anticommutation relations with all but the following anticommutators vanishing:

$$\{b_\beta(t_i), b^\dagger_{\beta'}(t_i)\} = \delta(\beta,\beta') \ ,$$

$$\{d_\beta(t_i), d^\dagger_{\beta'}(t_i)\} = \delta(\beta,\beta') \ , \quad \text{for } i = 1,2. \qquad (1.3)$$

Substitution of (1.1) and (1.2) into (1.3) yields:

$$\int d\alpha M_1(\gamma,\alpha)M_1^*(\gamma',\alpha) + \int d\beta M_2(\gamma,\beta)M_2^*(\gamma',\beta) = \delta(\gamma,\gamma'),$$

$$\int d\alpha M_3(\lambda,\alpha)M_3^*(\lambda',\alpha) + \int d\beta M_4(\lambda,\beta)M_4^*(\lambda',\beta) = \delta(\lambda,\lambda'),$$

$$\int d\alpha M_1(\gamma,\alpha)M_3^*(\lambda,\alpha) + \int d\beta M_2(\gamma,\beta)M_4^*(\lambda,\beta) = 0 , \qquad (1.4)$$

and

$$\int d\gamma M_1(\gamma,\alpha)M_1^*(\gamma,\alpha') + \int d\lambda M_3(\lambda,\alpha)M_3^*(\lambda,\alpha') = \delta(\alpha,\alpha'),$$

$$\int d\gamma M_2(\gamma,\beta)M_2^*(\gamma,\beta') + \int d\lambda M_4(\lambda,\beta)M_4^*(\lambda,\beta') = \delta(\beta,\beta'),$$

$$\int d\gamma M_1(\gamma,\alpha)M_2^*(\gamma,\beta) + \int d\lambda M_3(\lambda,\alpha)M_4^*(\lambda,\beta) = 0 . \qquad (1.5)$$

<u>The Integral Equations</u>

Consider a system which is in the vacuum state at time t_1. To determine what creation has occurred between times t_1 and t_2, $|0(t_1)>$ is expressed in terms of physical particle states, as defined at time t_2:

$$|0(t_1)> = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\sqrt{n!m!}} \int d\underline{\alpha}_n d\underline{\beta}_m \psi^{nm}(\underline{\alpha}_n,\underline{\beta}_m) b^\dagger_{\alpha_1}(t_2) \ldots\ldots d^\dagger_{\beta_m}(t_2)|0(t_2)> . \qquad (1.6)$$

The coefficients ψ^{nm} are to be evaluated. By definition:

$$b(t_1)|0(t_1)> = d(t_1)|0(t_1)> = 0 . \qquad (1.7)$$

Restating (1.7) using (1.6) and (1.2)

$$\int d\alpha_j M_1^*(\alpha_j,\gamma)\psi^{nm}(\underline{\alpha}_n,\underline{\beta}_m) = \sqrt{nm} \sum_{i=1}^{n} (-1)^{n+i+j} M_3^*(\beta_i,\gamma) \quad \times$$

$$\psi^{n-1\ m-1}\left(\frac{\alpha_n}{\hat{j}},\frac{\beta_m}{\hat{i}}\right) \tag{1.8}$$

$$\int d\beta_j M_4(\beta_j,\lambda)\psi^{nm}(\underline{\alpha}_n,\underline{\beta}_m) = \sqrt{nm} \sum_{i=1}^{n} (-1)^{n+i+j+1} M_2(\alpha_i,\lambda) \quad \times$$

$$\psi^{n-1\ m-1}\left(\frac{\alpha_n}{\hat{i}},\frac{\beta_m}{\hat{j}}\right) \tag{1.9}$$

Many properties of the integral equations (1.8) and (1.9) were investigated by G. Labonté [1973]. These abbreviations are used:

$$\int d\alpha M_1^*(\alpha,\gamma)\psi(\alpha) = \phi(\gamma), \quad \text{or} \quad M_1^*\psi = \phi \tag{1.8'}$$

and

$$\int d\beta M_4(\beta,\lambda)\Pi(\beta) = \chi(\lambda), \quad \text{or} \quad M_4\Pi = \chi\ . \tag{1.9'}$$

The possible forms of $\psi(\alpha)$ and $\Pi(\beta)$ are sought.

In the event that $\int d\gamma M_1(\alpha,\gamma)\phi(\gamma) = 0$, then $\phi(\gamma) = 0$. If $\int d\gamma M_1(\alpha,\gamma)\phi(\gamma) \neq 0$, then $\psi(\alpha)$ may sometimes exist. Define $F_n(\gamma)$ $(n = 1,\ldots.M)$ as the solutions of

$$\int d\gamma M_1(\alpha,\gamma)F(\gamma) = 0\ , \tag{1.10}$$

or equivalently $\int d\beta M_3^*(\beta,\gamma)\int d\gamma' M_3(\beta,\gamma')F(\gamma') = F(\gamma)$. It is possible that $M = 0$. With these definitions $\psi(\alpha)$ exists if and only if $\phi(\gamma)$ is orthogonal to $F_n(\gamma)$ for all n and in that case:

$$\psi = \sum_{r=0}^{\infty} (M_2 M_2^*)^r M_1 \phi + \phi_o , \tag{1.11}$$

where

$$(I - M_2 M_2^*)\phi_o = 0 .$$

Similar results hold for $\Pi(\beta)$ and $\chi(\lambda)$. If $\int d\lambda M_4^*(\beta,\lambda)\chi(\lambda) = 0$, then $\chi(\lambda) = 0$. If $\int d\lambda M_4^*(\beta,\lambda)\chi(\lambda) \neq 0$, then $\Pi(\beta)$ exists if and only if $\chi(\lambda)$ is orthogonal to all $F_n'(\lambda)$ $(n = 1,\ldots.N)$. The $F_n'(\lambda)$ satisfy:

$$\int d\alpha M_2(\alpha,\lambda)\int d\lambda' M_2^*(\alpha,\lambda')F'(\lambda') = F'(\lambda)$$

or equivalently $\int d\lambda M_4^*(\beta,\lambda)F'(\lambda) = 0$. (1.12)

When $\Pi(\beta)$ does exist, it has the form:

$$\Pi = \sum_{r=0}^{\infty} (M_3^* M_3)^r M_4^* \chi + \chi_o , \tag{1.13}$$

where

$$(I - M_3^* M_3)\chi_o = 0 .$$

There is a one to one correspondence between the M solutions of:

$$\int d\beta M_4(\beta,\lambda)G(\beta) = 0$$

and $\int d\gamma M_1(\alpha,\gamma)F(\gamma) = 0$. (1.14)

There is also a one to one correspondence between the N solutions of:

$$\int d\alpha M_1^*(\alpha,\gamma)G'(\alpha) = 0$$

and $$\int d\lambda M_4^*(\beta,\lambda)F'(\lambda) = 0 \ . \tag{1.15}$$

These G and G' functions are the quantities that will appear in ψ^{nm}.

Much algebraic consideration is needed to finish solving (1.8) and (1.9). Most $\psi^{nm}(\underline{\alpha}_n,\underline{\beta}_m)$ are zero. Only $\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell},\underline{\beta}_{M+\ell})$ survive, $\ell = 0,1,\ldots$.

$$\psi^{NM}(\underline{\alpha}_N,\underline{\beta}_M) = c \det \begin{pmatrix} G'(\underline{\alpha}_N) & 0 \\ 0 & G(\underline{\beta}_M) \end{pmatrix} , \tag{1.16}$$

$$\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell},\underline{\beta}_{M+\ell}) = \frac{c(-1)^{\frac{\ell(\ell+1)}{2}}}{\sqrt{(N+\ell)!(M+\ell)!}} \times$$

$$\times \det \left(\begin{array}{c|ccc} G'(\underline{\alpha}_{N+\ell}) & \begin{matrix} \chi(\alpha_1,\beta_1) & \ldots\ldots & \chi(\alpha_1,\beta_{M+\ell}) \\ \vdots & & \vdots \\ \chi(\alpha_{N+\ell},\beta_1) & \ldots\ldots & \chi(\alpha_{N+\ell},\beta_{M+\ell}) \end{matrix} \\ \hline 0 & G(\underline{\beta}_{M+\ell}) \end{array}\right) \tag{1.17}$$

where

$$\chi(\alpha_j,\beta_i) = \sum_{r=0}^{\infty}\int d\alpha\,(M_2M_2^*)^r(\alpha_j,\alpha)\int d\gamma M_1(\alpha,\gamma)M_3^*(\beta_i,\gamma) \ . \tag{1.18}$$

So it is seen that knowledge of the Bogoliubov transformations (1.1) and (1.2) allows calculation of the desired creation amplitudes.

2) Some Properties of the Creation Amplitudes

The probability amplitude for creating $N+\ell$ electrons and $M+\ell$ positrons with quantum numbers $\underline{\alpha}_{N+\ell}$ and $\underline{\beta}_{M+\ell}$ respectively is given by (1.17). In this situation N electrons are created in isolation, M positrons are created in isolation and ℓ electron-positron pairs are also created.

That $\psi^{N+\ell,M+\ell}$ is the determinant of a matrix guarantees that fermion statistics are in use. There are many ways to choose N electrons from $N+\ell$ to be created in isolation and many ways to choose M positrons from $M+\ell$ to be created in isolation. The remaining ℓ electrons and ℓ positrons may be matched off in pairs in many ways. Each of these many ways of creating the $N+\ell$ electrons with quantum numbers $\underline{\alpha}_{N+\ell}$ and the $M+\ell$ positrons with quantum numbers $\underline{\beta}_{M+\ell}$ contributes with equal probability amplitude to $\psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell},\underline{\beta}_{M+\ell})$.

The form of (1.17) further demonstrates that each of the creation events, whether isolated or pair, remains independent of the others. This is exactly what is expected in an external field problem: it is impossible for the electrons and positrons to interact so creation of one particle or pair doesn't effect any other creation.

There are the two types of interaction of the external field with the particle field. The first type is creation or annihilation of a pair or scattering of a particle. The second type is isolated creation or annihilation of a particle. These may be illustrated with "Feynman"[2)] diagrams (figure 1).

Since all creation events are just a mixture of isolated particle creation and pair creation events, the amplitudes G, G' and χ contain the entirety of information concerning creation. If the external field is weak, in the sense that $N = M = 0$, it is the function χ alone which is of interest.

3) Creation from Particle States

In sections 1) and 2), the system was in the vacuum state at time t_1. If the system were in some particle state at time t_1 no essential difference is expected because the particles don't interact with each other.

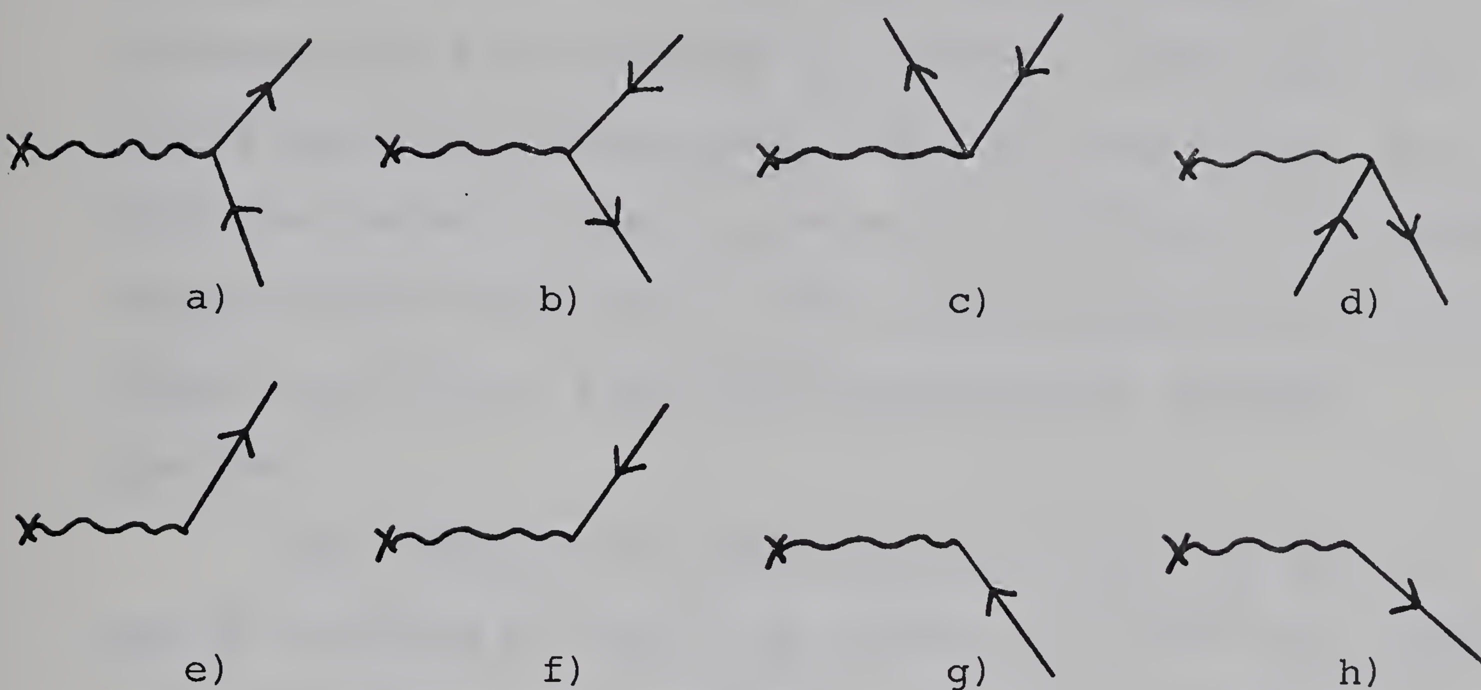
Suppose the system contains one electron of type α at time t_1. The physical appearance of the system at time t_2 may be obtained by applying (1.2) to (1.6):

$$b_\alpha^\dagger(t_1)|0(t_1)> = \sum_{\ell=0}^{\infty} \frac{1}{\sqrt{(N+\ell)!(M+\ell)!}} \int d\underline{\alpha}_{N+\ell} d\underline{\beta}_{M+\ell} \times$$

$$\times \psi^{N+\ell,M+\ell}(\underline{\alpha}_{N+\ell},\underline{\beta}_{M+\ell}) \times$$

$$\times\left(\int d\gamma M_1(\gamma,\alpha) b_\gamma^\dagger(t_2) + \int d\lambda M_3(\lambda,\alpha) d_\lambda(t_2)\right) b\underline{\alpha}_{N+\ell}^\dagger(t_2) d\underline{\beta}_{M+\ell}^\dagger |0(t_2)>. \tag{3.1}$$

Fig. 1 Possible "Feynman" Diagrams



The eight basic "Feynman" diagrams for the interaction of electrons and positrons with an external field: a) electron scattering, b) positron scattering, c) pair creation, d) pair annihilation, e) isolated electron creation, f) isolated positron creation, g) isolated electron annihilation, and h) isolated positron annihilation.

Consider the ℓth term in this summation. This is a superposition of two states: the first contains $N+\ell+1$ electrons and $M+\ell$ positrons as defined at time t_2; the second contains $N+\ell$ electrons and $M+\ell-1$ positrons. As with the vacuum at time t_1, here $N+\ell$ electrons are created along with $M+\ell$ positrons. The initially present electron simply survives or else annihilates with a created positron.

More complicated many particle states at time t_1 may be examined at time t_2 by using (1.2) and (1.6) again. No new phenomena are observed - creation is completely described by the $\psi^{N+\ell,M+\ell}$ found in (1.17) and by the relevant Bogoliubov transformation (1.2).

4) Weak External Fields

The situation of $N = M = 0$ will be easiest to handle since only pair creation occurs. A Bogoliubov transformation where $N = M = 0$ is called weak and the corresponding external field is also referred to as weak. The following situation helps motivate the terminology "weak" and illustrates that the class of weak external fields is not devoid of interest.

Consider an external field $A(x) = \theta(t-t_o)A(\underset{\sim}{x})$ which satisfies:

$$|(\psi, e\gamma^{o}\gamma\cdot A\psi)| < (\psi, |h_{o}|\psi) \qquad \forall\, \psi \in [\mathcal{L}^{2}(R^{3})]^{4}. \qquad (4.1)$$

The potential energy of any particle field in this external field is strictly less than its kinetic energy. T. Kato [1966] has discussed such fields and proved that (4.1) guarantees the essential self adjointness of the c-number operator h on a large domain. Many useful potentials (e.g. simple harmonic oscillator and Coulomb for $z \leq 87$) satisfy (4.1).

The values of N and M for this field are sought. How many solutions are there to the following equations?

$$\int d\gamma M_{1}(\alpha,\gamma)F(\gamma) = 0 \ , \qquad (4.2)$$

$$\int d\lambda M_{4}^{*}(\beta,\lambda)F'(\lambda) = 0 \ . \qquad (4.3)$$

Using (II-2.8) with $t_{1} = t_{o}$ and $t > t_{o}$, these equations become:

$$\sum_{\gamma} (f_{+\alpha}, e^{-ih(t-t_{o})} f^{o}_{+\gamma})F(\gamma) = 0 \ , \qquad (4.4)$$

$$\sum_{\lambda} (f_{-\beta}, e^{-ih(t-t_{o})} f^{o}_{-\lambda})F'^{*}(\lambda) = 0 \ , \qquad (4.5)$$

where $f_{\varepsilon\alpha}$ is a c-number wave function corresponding to the full operator $h = h_{o} - e\gamma^{o}\gamma\cdot A(\underset{\sim}{x})$. Since the $f_{\varepsilon\alpha}$ are eigenfunctions of h, (4.4) and (4.5) are equivalent to:

$$\sum_{\gamma} (f_{+\alpha}, f^{o}_{+\gamma})F(\gamma) = 0 \ , \qquad (4.6)$$

$$\sum_{\lambda} (f_{-\beta}, f^{o}_{-\lambda})F'^{*}(\lambda) = 0 \ . \qquad (4.7)$$

For (4.6) and (4.7) to hold, sets of coefficients $\{a_\beta\} \neq \{0\}$ and $\{b_\delta\} \neq \{0\}$ must exist such that:

$$\psi_+^o = \sum_\gamma f_{+\gamma}^o)F(\gamma) = \sum_\beta f_{-\beta})a_\beta \quad , \tag{4.8}$$

$$\psi_-^o = \sum_\lambda f_{-\lambda}^o)F'^*(\lambda) = \sum_\delta f_{+\delta})b_\delta \quad . \tag{4.9}$$

The following matrix elements are of interest:

$$\begin{aligned}(\psi_-^o, h\psi_-^o) &= (\psi_-^o,\ h_o\psi_-^o) + (\psi_-^o, (-e\gamma^o\gamma\cdot A)\psi_-^o) \\ &= -(\psi_-^o, |h_o|\psi_-^o) + (\psi_-^o, (-e\gamma^o\gamma\cdot A)\psi_-^o) \\ &\leq -(\psi_-^o, |h_o|\psi_-^o) + |(\psi_-^o, e\gamma^o\gamma\cdot A\psi_-^o)| \ . \\ &< 0 \quad ,\end{aligned} \tag{4.10}$$

but:

$$(\psi_-^o, h\psi_-^o) = \sum_\delta \sum_{\delta'} b_{\delta'}^* b_\delta (f_{+\delta'}, hf_{+\delta}) \geq 0 \quad . \tag{4.11}$$

Also:

$$\begin{aligned}(\psi_+^o, h\psi_+^o) &= (\psi_+^o, h_o\psi_+^o) + (\psi_+^o, (-e\gamma^o\gamma\cdot A)\psi_+^o) \\ &= (\psi_+^o, |h_o|\psi_+^o) + (\psi_+^o, (-e\gamma^o\gamma\cdot A)\psi_+^o) \ . \\ &> 0 \ ,\end{aligned} \tag{4.12}$$

but:

$$(\psi_+^o, h\psi_+^o) = \sum_\beta \sum_{\beta'} a_{\beta'}^* a_\beta (f_{-\beta'}, hf_{-\beta}) \leq 0 \quad . \tag{4.13}$$

The contradiction between (4.10) and (4.11) and between (4.12) and (4.13) indicates that no non-trivial solution exists for either (4.2) or (4.3). In other words, $N = M = 0$ and the external field is weak.

5) <u>Strong External Fields</u>

Consider now an external field for which at least one of N and M is non-zero. From the discussion of sections 2) and 3), it is known that N isolated electrons and M isolated positrons are created with certainty.

If $N \neq M$, then charge is not conserved. The charge operator at time t is

$$Q(t) = e \sum_{\beta} (b^{\dagger}_{\beta}(t) b_{\beta}(t) - d^{\dagger}_{\beta}(t) d_{\beta}(t)) \ . \qquad (5.1)$$

Applying $Q(t_1)$ and $Q(t_2)$ to the Fock basis of states at either time t_1 or t_2 yields:

$$Q(t_2) = Q(t_1) + (N-M)eI \ . \qquad (5.2)$$

While $[Q(t),H(t)] = 0$ at all times, charge is not conserved. This is a result of $Q(t)$'s <u>explicit</u> dependence on time through $b_{\beta}(t)$ and $d_{\beta}(t)$.

The calculations of section 4) give some insight into isolated creation. A positron may be created in isolation ($M \neq 0$) only if a non-trivial solution exists for (4.2). Examining (4.12) and (4.13) this may happen only if:

$$(\psi_+^o, h_o\psi_+^o) + (\psi_+^o, -e\gamma^o\gamma\cdot A\psi_+^o) \leq 0 \ . \tag{5.3}$$

In other words, the total energy of the field ψ_+^o must become negative after the large potential energy $-e\gamma^o\gamma\cdot A$ has been added to the system at time t_o. While ψ_+^o contains only free electron states before t_o, it must contain only positron states after t_o - an electron energy level has "crossed" over $E = 0$ and become a positron level.

In the case of isolated positron creation, equations (1.1), (1.14) and (4.2) indicate that for an appropriate λ:

$$d_\lambda^\dagger(t_2) = \int d\alpha M_3(\lambda,\alpha) b_\alpha(t_1) \ . \tag{5.4}$$

To see the effect of such a Bogoliubov transformation on charge conservation consider this example:

$$d^\dagger(t_2) = b(t_1) \tag{5.5}$$

and compute the charge contribution from this state.

$$\begin{aligned} Q(t_1) &= eb^\dagger(t_1)b(t_1) \\ &= ed(t_2)d^\dagger(t_2) \\ &= e(1 + d^\dagger(t_2)d(t_2)) \\ &= e + Q(t_2) \ . \end{aligned} \tag{5.6}$$

The strong Bogoliubov transformation has shifted an electron level into the negative energy region. An isolated

positron was created and an extra charge of -e, the charge of a positron, was observed.

6) Pair Creation Amplitude

The External Field and Its Auxiliaries

The pair creation amplitude $\chi(\alpha,\beta)$ will be examined for a particular class of external fields $A(\underset{\sim}{x},t)$. Only weak external fields which have no bound states are considered. Furthermore, since a scattering experiment is to be performed, A must have good behaviour for large times:

$$\lim_{t\to-\infty} A(\underset{\sim}{x},t) = A^{in}(\underset{\sim}{x}) \ ,$$

$$\lim_{t\to+\infty} A(\underset{\sim}{x},t) = A^{out}(\underset{\sim}{x}) \ . \tag{6.1}$$

The spacial dependence of A^{in} and A^{out} must allow asymptotically free states according to criterion (II-1.13).

If $f^{in}_{\varepsilon\beta}$ and $f^{out}_{\varepsilon\beta}$ are c-number wave functions corresponding to A^{in} and A^{out} respectively, the Bogoliubov matrices are:

$$M_1(\beta,\beta') = (f^{out}_{+\beta},u(\infty,-\infty)f^{in}_{+\beta'}) \ ,$$

$$M_2(\beta,\beta') = (f^{out}_{+\beta},u(\infty,-\infty)f^{in}_{-\beta'}) \ ,$$

$$M_3(\beta,\beta') = (f^{out}_{-\beta},u(\infty,-\infty)f^{in}_{+\beta'}) \ ,$$

$$M_4(\beta,\beta') = (f^{out}_{-\beta},u(\infty,-\infty)f^{in}_{+\beta'}) \ , \tag{6.2}$$

with

$$f^{in}_{\varepsilon\beta} = \Omega_{+}(A^{in})f^{o}_{\varepsilon\beta} \;,$$

$$f^{out}_{\varepsilon\beta} = \Omega^{\dagger}_{-}(A^{out})f^{o}_{\varepsilon\beta} \;.$$

From Chapter II, it is known that time independent external fields don't cause creation. Scattering amplitudes for change of momentum processes are given by (II-1.24). Without much loss of generality, A^{in} and A^{out} are chosen to be zero. As a result, $f^{in}_{\varepsilon\beta} = f^{out}_{\varepsilon\beta} = f^{o}_{\varepsilon\beta}$ and equations (6.2) become simpler.

The matrices of (6.2) may be obtained from limits of a number of different external field problems. Consider firstly this potential:

$$\begin{aligned} A'(x) &= A(\underset{\sim}{x},t_1), && t < t_1 \\ A'(x) &= A(x) \;, && t_1 < t < t_2 \\ A'(x) &= A(\underset{\sim}{x},t_2), && t > t_2 \end{aligned}$$

with c-number wave functions:

$$f^{1}_{\varepsilon\beta}(\underset{\sim}{x}) = \Omega_{+}(A(\underset{\sim}{x},t_1))f^{o}_{\varepsilon\beta} \quad ,$$

and

$$f^{2}_{\varepsilon\beta}(\underset{\sim}{x}) = \Omega^{\dagger}_{-}(A(\underset{\sim}{x},t_2))f^{o}_{\varepsilon\beta} \quad . \qquad (6.3)$$

This is the auxiliary field problem as set up in (II-2). The Bogoliubov transformation relates operators at time t_1 to those at time t_2 by:

$$M'_1(\beta,\beta') = (f^{2}_{+\beta},u(t_2,t_1)f^{1}_{+\beta'}) \;, \text{ etc.} \qquad (6.4)$$

Taking $t_1 \to -\infty$ and $t_2 \to +\infty$ in (6.4) reproduces (6.2).

A second related external field problem is:

$$A''(x) = 0 \quad , \qquad t < t_1$$
$$A''(x) = A(x) \quad , \qquad t_1 < t < t_2$$
$$A''(x) = 0 \quad , \qquad t > t_2 \quad . \tag{6.5}$$

The corresponding Bogoliubov transformation is:

$$M_1''(\beta,\beta') = (f^o_{+\beta}, u(t_2,t_1) f^o_{+\beta'}) \text{ , etc.} \tag{6.6}$$

Taking $t_1 \to -\infty$ and $t_2 \to +\infty$ in (6.6) also reproduces (6.2). Switching the potential on suddenly at time t_1 and off suddenly at time t_2 may introduce Fourier components into the decomposition of A" which aren't present in A. However, since A becomes small as $t_1 \to -\infty$ and $t_2 \to +\infty$, the jump which A" suffers at times t_1 and t_2 becomes small and vanishes in the limit. In this limit, A, A' and A" have same frequency components.

To calculate $\chi(\alpha,\beta)$, the Bogoliubov transformation (6.6) will be used and the limits $t_1 \to -\infty$ and $t_2 \to +\infty$ taken. For convenience, the primes on the M_i in (6.6) will be omitted. As mentioned above, for this sytem $\chi(\alpha,\beta)$ will contain the complete knowledge of creation phenomena.

Explicit Calculation of $\chi(\alpha,\beta)$

The representation of the Dirac γ matrices which appears under Notation is used. The definitions of $f^o_{\varepsilon\beta}$, $w^s(\underset{\sim}{p})$ and $v^s(\underset{\sim}{p})$ of (II-1.5) and (II-1.6) are retained.

In the standard representation:

$$w^s(\underset{\sim}{p}) = \sqrt{\frac{\omega(p)+m}{2m}}\left(I - \frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m}\right) w^s(0) \ ,$$

$$v^s(\underset{\sim}{p}) = \sqrt{\frac{\omega(p)+m}{2m}}\left(I + \frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m}\right) v^s(0) \ , \tag{6.7}$$

where

$$w^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

$$w^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ,$$

$$v^1(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ,$$

$$v^2(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Remembering that

$$(f,g) = \int d^3x \ f^{\dagger}(\underset{\sim}{x})g(\underset{\sim}{x})$$

the orthonormality relations follow:

$$(f^{o}_{\varepsilon \underset{\sim}{p} s}, f^{o}_{\varepsilon' \underset{\sim}{p}' s'}) = \delta_{\varepsilon\varepsilon'}\delta_{ss'}\delta(\underset{\sim}{p}-\underset{\sim}{p}') \ . \tag{6.8}$$

The Fourier transform of the c-number time translation operator is defined:

$$\tilde{u}(\underset{\sim}{p};\underset{\sim}{q}) = \frac{1}{(2\pi)^{3/2}} \int d^3x \ e^{i\underset{\sim}{p}.\underset{\sim}{x}} u(t_2,t_1) e^{-i\underset{\sim}{q}.\underset{\sim}{x}} \ . \tag{6.9}$$

Since $u(t_2,t_1)$ contains $\underset{\sim}{\partial}$ dependence as well as the $\underset{\sim}{x}$ dependence which it inherited from $A(x)$, it doesn't commute with c-number functions such as $e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}$.

It is possible to calculate (6.6) in detail:

$$M_1(\underset{\sim}{p}r;\underset{\sim}{q}s) = \frac{1}{(2\pi)^{3/2}} \frac{m}{\sqrt{\omega(p)\omega(q)}} w^r(\underset{\sim}{p})^\dagger \tilde{u}(\underset{\sim}{p};\underset{\sim}{q}) w^s(\underset{\sim}{q})$$

$$M_2(\underset{\sim}{p}r;\underset{\sim}{q}s) = \frac{1}{(2\pi)^{3/2}} \frac{m}{\sqrt{\omega(p)\omega(q)}} w^r(\underset{\sim}{p})^\dagger \tilde{u}(\underset{\sim}{p};-\underset{\sim}{q}) v^s(\underset{\sim}{q})$$

$$M_3(\underset{\sim}{p}r;\underset{\sim}{q}s) = \frac{1}{(2\pi)^{3/2}} \frac{m}{\sqrt{\omega(p)\omega(q)}} v^r(\underset{\sim}{p})^\dagger \tilde{u}(-\underset{\sim}{p};\underset{\sim}{q}) w^s(\underset{\sim}{q})$$

$$M_4(\underset{\sim}{p}r;\underset{\sim}{q}s) = \frac{1}{(2\pi)^{3/2}} \frac{m}{\sqrt{\omega(p)\omega(q)}} v^r(\underset{\sim}{p})^\dagger \tilde{u}(-\underset{\sim}{p};-\underset{\sim}{q}) v^s(\underset{\sim}{q}) . \tag{6.10}$$

Using:

$$\sum_s w^s(\underset{\sim}{p})^\dagger w^s(\underset{\sim}{p}) = \frac{\gamma\cdot p+m}{2m}\gamma^o , \tag{6.11}$$

it follows that:

$$M_1M_3^*(\underset{\sim}{p}s\underset{\sim}{q}r) = \frac{1}{(2\pi)^3} \frac{m^2}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \left\{\int \frac{d^3P}{\omega(P)} \tilde{u}(\underset{\sim}{p};\underset{\sim}{P}) \times \right.$$

$$\left. \times \frac{\gamma\cdot P+m}{2m} \gamma_o [\tilde{u}(-\underset{\sim}{q};\underset{\sim}{P})]^\dagger \right\} v^r(\underset{\sim}{q})$$

$$= \frac{m\sqrt{(\omega(p)+m)\,(\omega(q)+m)}}{2(2\pi)^3 \sqrt{\omega(p)\omega(q)}} w^s(0)^\dagger \left\{\int \frac{d^3P}{\omega(P)} \left(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m}\right)\tilde{u}(\underset{\sim}{p};\underset{\sim}{P}) \frac{\gamma\cdot P+m}{2m}\gamma_o \times \right.$$

$$\left. \times [\tilde{u}(-\underset{\sim}{q};\underset{\sim}{P})]^\dagger \left(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{q}}{\omega(q)+m}\right)\right\} v^r(0) . \tag{6.12}$$

Replacing $\tilde{u}(\underset{\sim}{p};\underset{\sim}{P})$ and $\tilde{u}(\underset{\sim}{q};\underset{\sim}{P})$ by their defining integrals, another expression for (6.12) is obtained:

$$M_1M_3^*(\underset{\sim}{p}s\underset{\sim}{q}r) = \frac{m\sqrt{(\omega(p)+m)(\omega(q)+m)}}{2(2\pi)^6\sqrt{\omega(p)\omega(q)}} w^s(0)^\dagger\{(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m})\int d^3x\, d^3y$$

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}u(\underset{\sim}{x})[\int \frac{d^3P}{\omega(P)} e^{-i\underset{\sim}{P}\cdot\underset{\sim}{x}} \frac{\gamma\cdot P+m}{2m}\gamma_o e^{i\underset{\sim}{P}\cdot\underset{\sim}{y}}]u^\dagger(\underset{\sim}{y})e^{i\underset{\sim}{q}\cdot\underset{\sim}{y}}$$

$$(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{q}}{\omega(q)+m})\}v^r(0)\ . \tag{6.13}$$

The P integration may be carried out explicitly:

$$M_1M_3^*(\underset{\sim}{p}s\underset{\sim}{q}r) = \frac{m\sqrt{(\omega(p)+m)(\omega(q)+m)}}{2(2\pi)^6\sqrt{\omega(p)\omega(q)}} w^s(0)^\dagger\{(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m})\int d^3x\, d^3y$$

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}u(\underset{\sim}{x})F(\underset{\sim}{x}-\underset{\sim}{y})u^\dagger(\underset{\sim}{y})e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}(I+\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{q}}{\omega(q)+m})\}v^r(0)\,, \tag{6.14}$$

where

$$F(\underset{\sim}{z}) = \frac{(2\pi)^3}{2m}\delta(\underset{\sim}{z}) + \frac{2\pi m}{z}\gamma^o(-i\gamma\cdot\hat{z}K_2(mz) + K_1(mz))\ . \tag{6.15}$$

By (1.18), the pair creation amplitude χ vanishes if $M_1M_3^*$ does, so it is hoped that (6.12) and (6.14) are zero when the field A is not suitable for creating pairs. The only unknown quantities in (6.12) and (6.14) are u and $\tilde{u}$. It is the investigation of the properties of these operators that is of concern in Chapters IV and V.

CHAPTER IV

A PERTURBATION CALCULATION

1) An Integral Equation

In this section an integral equation for the c-number time translation operator u will be found. This equation will be useful for obtaining a series expansion of u in powers of the coupling constant e.

The time translation operator $u(t_2,t_1)$ satisfies:

$$i \frac{\partial}{\partial t_2} u(t_2,t_1) = (h_o + h'(t_2))u(t_2,t_1), \qquad (1.1a)$$

and

$$-i \frac{\partial}{\partial t_1} u(t_2,t_1) = u(t_2,t_1)(h_o + h'(t_1)), \qquad (1.1b)$$

where $h'(t) = -e\gamma^o\gamma\cdot A(\underset{\sim}{x},t)$. (1.2)

The following initial condition holds

$$u(t,t) = 1 , \quad \forall\, t . \qquad (1.3)$$

The free field time translation u_o satisfies (1.1) and (1.3) with $A(x) \equiv 0$:

$$u_o(t_2,t_1) = e^{-i(t_2-t_1)h_o} . \qquad (1.4)$$

Equation (1.1a) suggests that

$$u(t_2,t_1) = u_o(t_2,t_1) + \int G(t_2,t)h'(t)u(t,t_1)dt , \qquad (1.5)$$

where $(i \frac{\partial}{\partial t_2} - h_o)G(t_2,t) = \delta(t_2-t)$. (1.6)

Consideration of (1.1b) and (1.3) allows selection of the appropriate Green's function G. Equation (1.5) becomes:

$$u(t_2,t_1) = u_o(t_2,t_1) - i \int_{t_1}^{t_2} e^{-i(t_2-t)h_o} h'(t)u(t,t_1)dt \ . \tag{1.7}$$

2) <u>The Perturbation Expansion of u</u>

Equation (1.7) may be rewritten to show the occurrence of the coupling constant e explicitly:

$$u(t_2,t_1) = u_o(t_2,t_1) + ie \int_{t_1}^{t_2} e^{-i(t_2-t)h_o} \gamma^o \gamma \cdot A(\underset{\sim}{x},t)u(t,t_1)dt. \tag{2.1}$$

In (2.1), $u(t,t_1)$ may be rewritten using (2.1) itself:

$$u(t_2,t_1) = u_o(t_2,t_1) + ie \int_{t_1}^{t_2} e^{-i(t_2-\tau)h_o} \gamma^o \gamma \cdot A(\underset{\sim}{x},\tau)u_o(\tau,t_1)d\tau$$

$$-e^2 \int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' e^{-i(t_2-\tau)h_o} \gamma^o \gamma \cdot A(\underset{\sim}{x},\tau) e^{-i(\tau-\tau')h_o} \times$$

$$\gamma^o \gamma \cdot A(\underset{\sim}{x},\tau')u(\tau',t_1) \tag{2.2}$$

This process of replacing each occurrence of u on the right side of the integral equation by the formula (2.1) may be repeated again and again:

$$u(t_2,t_1) = \sum_{i=0}^{n} (-e)^i u_i(t_2,t_1) + (-e)^{n+1} \int_{t_1}^{t_2=\tau_{n+2}} d\tau_{n+1} \cdots$$

$$\int_{t_1}^{\tau_2} d\tau_1 \prod_{i=1}^{n+1} (-ie^{-i(\tau_{i+1}-\tau_i)h_o} \gamma^o \gamma \cdot A(\underset{\sim}{x},\tau_i)) u(\tau_1,t_1) , \tag{2.3}$$

where

$$u_j(t_2,t_1) = \int_{t_1}^{t_2=\tau_{j+1}} d\tau_j \cdots \int_{t_1}^{\tau_2} d\tau_1 \prod_{i=1}^{j} (-ie^{-i(\tau_{i+1}-\tau_i)h_o} \times$$

$$\gamma^o \gamma \cdot A(\underset{\sim}{x},\tau_i)) u_o(\tau_1,t_1) . \tag{2.4}$$

In this manner an expansion for u is obtained in terms of powers of the coupling constant e. Notice that after n iterations, u_o appears in the first n terms and it is only in the term of order e^{n+1}, that u is found. If A(x) is known, it is possible to calculate these first n terms explicitly. Since $e^2 \sim 1/137$, it is possible that the formal expansion (2.5) for u obtained from infinitely many iterations may converge:

$$u(t_2,t_1) = \sum_{i=0}^{\infty} (-e)^i u_i(t_2,t_1) . \tag{2.5}$$

The series (2.5) is the Neumann series for u. In the case that this series converges, u may be computed term by term from known quantities and may be meaningfully approximated by truncation of the series.

3) Perturbation Expansion of the Pair Creation Amplitude

Since the Bogoliubov matrices are matrix elements of u, equation (2.5) may be used to obtain a perturbation expansion for these matrices.

$$M_i(\beta,\beta') = \sum_{n=0}^{\infty} (-e)^n M_i^{(n)}(\beta,\beta'), \qquad (3.1)$$

where

$$M_1^{(n)}(\beta,\beta') = (f^o_{+\beta}, u_n f^o_{+\beta'}),$$

$$M_2^{(n)}(\beta,\beta') = (f^o_{+\beta}, u_n f^o_{-\beta'}), \text{ etc.} \qquad (3.2)$$

The matrix element $(f_{\varepsilon\beta}, u_n f_{\varepsilon'\beta'})$ is the amplitude of a proper Feynman diagram of n vertices connecting an incoming particle $(\varepsilon'\beta')$ with an outgoing particle $(\varepsilon\beta)$. Later computation of this matrix element using intermediate states will demonstrate that this matrix element describes the propagation of a free particle with n interactions with the external field.

From the perturbation expansion (3.1) of the M_i and the definition (III-1.18) of the pair creation amplitude χ, an expansion for χ is found

$$\chi = \sum_{n=0}^{\infty} (-e)^n \chi^{(n)} , \qquad (3.3)$$

where

$$\chi^{(n)} = \sum_{i_m, j_m, r, k, \ell} \prod_{m=1}^{r} (M_2^{(i_m)} M_2^{(j_m)*})(M_1^{(k)} M_3^{(\ell)*}), \qquad (3.4)$$

with

$$\left(\sum_{m=1}^{r} i_m + j_m\right) + k + \ell = n \ .$$

The relations (3.2)-(3.4) provide a description of the "Feynman" diagram of Chapter III in terms of proper Feynman diagrams.

Zeroth Order

$$(f^o_{\varepsilon\beta}, u_o(t_2,t_1) f^o_{\varepsilon'\beta'}) = \delta_{\varepsilon\varepsilon'}\delta_{\beta\beta'} e^{-i(t_2-t_1)\varepsilon\omega(\beta)} \qquad (3.5)$$

Only M_1 and M_4 have non-vanishing zeroth order contributions. Using (3.4), $\chi^{(o)} = 0$ and the absence of pair creation in a free external field is reconfirmed.

First Order

$$(f^o_{\varepsilon\beta}, u_1(t_2,t_1) f^o_{\varepsilon'\beta'})$$

$$= -i\int d^3x \int_{t_1}^{t_2} d\tau \ f^{o\dagger}_{\varepsilon\beta}(\underset{\sim}{x}) e^{-i(t_2-\tau)h_o} \gamma^o \gamma\cdot A(\underset{\sim}{x},\tau) \, e^{-i(\tau-t_1)h_o} f^o_{\varepsilon'\beta'}(\underset{\sim}{x})$$

$$= -i\int d^3x \int_{t_1}^{t_2} d\tau \ f^{o\dagger}_{\varepsilon\beta}(\underset{\sim}{x}) e^{-i(t_2-\tau)\varepsilon\omega(\beta)} \gamma^o \gamma\cdot A(\underset{\sim}{x},\tau) \, e^{-i(\tau-t_1)\varepsilon'\omega(\beta')} f^o_{\varepsilon'\beta'}(\underset{\sim}{x}) . \qquad (3.6)$$

Letting $t_1 \to -\infty$ and $t_2 \to \infty$, (3.6) becomes (up to phase):

$$(f^o_{\varepsilon\beta}, u_1(\infty,-\infty) f^o_{\varepsilon'\beta'}) = (2\pi)^{\frac{1}{2}} \int d^3x \ f^{o\dagger}_{\varepsilon\beta}(\underset{\sim}{x}) \, \gamma^o \gamma\cdot\tilde{A}(\underset{\sim}{x}, \varepsilon\omega(\beta) - \varepsilon'\omega(\beta')) f^o_{\varepsilon'\beta'}(\underset{\sim}{x}) \ , \qquad (3.7)$$

where

$$\tilde{A}(\underset{\sim}{x},\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dt\, e^{i\omega t} A(\underset{\sim}{x},t) \quad . \tag{3.8}$$

The spatial integration gives (up to phase);

$$M_1^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi} \frac{m}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \gamma^o\gamma\cdot\tilde{\tilde{A}}(\underset{\sim}{q}-\underset{\sim}{p},\omega(p)-\omega(q))w^r(\underset{\sim}{q}),$$

$$M_2^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi} \frac{m}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \gamma^o\gamma\cdot\tilde{\tilde{A}}(-\underset{\sim}{q}-\underset{\sim}{p},\omega(p)+\omega(q))v^r(\underset{\sim}{q}),$$

$$M_3^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi} \frac{m}{\sqrt{\omega(p)\omega(q)}} v^s(\underset{\sim}{p})^\dagger \gamma^o\gamma\cdot\tilde{\tilde{A}}(\underset{\sim}{q}+\underset{\sim}{p},-\omega(p)-\omega(q))w^r(\underset{\sim}{q})$$

$$M_4^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi} \frac{m}{\sqrt{\omega(p)\omega(q)}} v^s(\underset{\sim}{p})^\dagger \gamma^o\gamma\cdot\tilde{\tilde{A}}(-\underset{\sim}{q}+\underset{\sim}{p},-\omega(p)+\omega(q))v^r(\underset{\sim}{q}), \tag{3.9}$$

where

$$\tilde{\tilde{A}}(\underset{\sim}{p},\omega) = \frac{1}{(2\pi)^2} \int d^4x\, e^{ip.x} A(x) \quad . \tag{3.10}$$

Notice that the Fourier transform, $\tilde{\tilde{A}}$, guarantees conservation of energy and momentum at the one vertex of interaction with the external field.

The first order contribution to pair creation is (up to phase):

$$\chi^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi} \frac{m}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \gamma^o\gamma\cdot\tilde{\tilde{A}}(-\underset{\sim}{p}\ -\underset{\sim}{q},+\omega(p)+\omega(q))v^r(\underset{\sim}{q}). \tag{3.11}$$

Unless the external field has a Fourier component corresponding to the total energy and momentum of the created pair, no creation occurs to this order.

<u>Higher Orders</u>

The formula (2.4) is rewritten:

$$u_n(t_2,t_1) = (-i)^n \int_{t_1}^{t_2=\tau_{n+1}} d\tau_n \cdots \int_{t_1}^{\tau_2} d\tau_1 \, e^{-it_2 h_o} \prod_{i=1}^{n} \times$$

$$\times \left(e^{i\tau_i h_o} \gamma^o \gamma \cdot A(\underset{\sim}{x},\tau_i) e^{-i\tau_i h_o}\right) e^{it_1 h_o} \quad . \tag{3.12}$$

Remembering that

$$\sum_{\varepsilon\beta} f^o_{\varepsilon\beta}(\underset{\sim}{x}) f^{o\dagger}_{\varepsilon\beta}(\underset{\sim}{y}) = \delta(\underset{\sim}{x}-\underset{\sim}{y}) \quad , \tag{3.13}$$

the matrix element of (3.12) may be obtained:

$$(f^o_{\varepsilon\beta}, u_n(t_2,t_1) f^o_{\varepsilon'\beta'}) = (-i)^n \sum_{\varepsilon_n\beta_n} \cdots \sum_{\varepsilon_2\beta_2} \int d^3x_n \cdots$$

$$\int d^3x_1 \int_{t_1}^{t_2=\tau_{n+1}} d\tau_n \cdots \int_{t_1}^{\tau_2} d\tau_1 \, e^{-it_2\varepsilon\omega(\beta)}$$

$$\prod_{i=1}^{n} \left(f^{o\dagger}_{\varepsilon_{i+1}\beta_{i+1}}(\underset{\sim}{x}_i) e^{i\tau_i\varepsilon_{i+1}\omega(\beta_{i+1})} \gamma^o \gamma \cdot A(\underset{\sim}{x}_i,\tau_i) \times\right.$$

$$\left.\times\, e^{-i\tau_i\varepsilon_i\omega(\beta_i)} f^o_{\varepsilon_i\beta_i}(\underset{\sim}{x}_i)\right) e^{it_1\varepsilon'\omega(\beta')} \quad , \tag{3.14}$$

where

$$(\varepsilon_{n+1},\beta_{n+1}) = (\varepsilon,\beta) \quad ,$$

and

$$(\varepsilon_1,\beta_1) = (\varepsilon',\beta') \quad .$$

Each of the spacial integrations may be done, producing a term proportional to a matrix element of the spacial Fourier transform of A. The argument of the Fourier transform gives momentum conservation term by term in the product in (3.14).

The time integrations are to be done in the limit $t_1 \to -\infty$ and $t_2 \to +\infty$. Using the abbreviation

$$A_i(\tau) = \int d^3x_i \; f^{o\dagger}_{\varepsilon_{i+1}\beta_{i+1}}(\underset{\sim}{x}_i)\gamma^o \gamma\cdot A(\underset{\sim}{x}_i,\tau) f^o_{\varepsilon_i\beta_i}(\underset{\sim}{x}_i), \quad (3.15)$$

(3.14) becomes (up to phase):

$$(f^o_{\varepsilon\beta}, u_n(\infty,-\infty) f^o_{\varepsilon'\beta'}) = \sum_{\varepsilon_n\beta_n} \cdots \sum_{\varepsilon_2\beta_2} \int_{-\infty}^{\infty} d\tau_n \int_{-\infty}^{\tau_n} d\tau_{n-1} \cdots \int_{-\infty}^{\tau_2} d\tau_1$$

$$\times \prod_{i=1}^{n} e^{i\tau_i(\varepsilon_{i+1}\omega(\beta_{i+1})-\varepsilon_i\omega(\beta_i))} A_i(\tau_i) \; . \quad (3.16)$$

Each $A_i(\tau_i)$ may be replaced by its Fourier decomposition and then the time integrations may be done:

$$(f^o_{\varepsilon\beta}, u_n(\infty,-\infty) f^o_{\varepsilon'\beta'}) = \frac{1}{(2\pi)^{\frac{n}{2}-1}} \sum_{\varepsilon_n\beta_n} \cdots \sum_{\varepsilon_2\varepsilon_2} \int_{-\infty}^{\infty} d\omega_n \cdots \int_{-\infty}^{\infty} d\omega_1$$

$$\times \delta(\varepsilon\omega(\beta)-\varepsilon'\omega(\beta')-\omega_n-\omega_{n-1}\cdots-\omega_1)\tilde{A}_n(\omega_n)$$

$$\times \prod_{i=1}^{n-1} \frac{\tilde{A}_i(\omega_i)}{i(\varepsilon_{i+1}\omega(\beta_{i+1})-\varepsilon'\omega(\beta')-\omega_i-\omega_{i-1}\cdots-\omega_1)} \; .$$

(3.17)

The nature of energy conservation in an n vertex Feynman diagram may be obtained from (3.17). The delta function insists on overall conservation of energy - any difference in energy between the initial and final wavefunctions must be made up by the n contributions from the external field. It is seen that there is not strict energy conservation at each vertex; however, while all values of ω_i do occur in $\tilde{A}_i(\omega_i)$, the energy denominator indicates that the strongest contribution comes from:

$$\omega_i \sim \varepsilon_{i+1}\omega(\beta_{i+1}) - \varepsilon_i\omega(\beta_i) \ . \qquad (3.18)$$

Because of the integrations over the ω_i, no threshold effect for pair creation emerges. Suppose $\tilde{A}_i(\omega) = 0$ if $|\omega| > 2m$ and examine (3.17) with $\varepsilon = -$ and $\varepsilon' = +$. It is possible for the sum of the ω_i to be larger than 2m even if each is smaller than 2m and so $M_3^{(n)}(\beta,\beta') = (f^o_{-\beta}, \dot{u}_n f^o_{+\beta'})$ may be nonzero and pair creation may occur.

It is important to understand where the non-conservation of energy at individual vertices arises. One of the outstanding features of a perturbation approach is the treatment of an interaction process step by step. Both in the above integral equation method and with Feynman diagrams, all time is divided

into ordered intervals - the potential during one interval has no effect on events in another. For this reason, the finite upper limits of integration appeared in (3.16) resulting eventually in the non-conservation of energy at individual vertices.

What remains to be determined is whether this non-conservation of energy is just a consequence of perturbation theory or whether it holds true for an exact solution. In the latter event no threshold effect would be present in pair creation and pairs could be produced by <u>any</u> time dependent external field.

CHAPTER V

THE c-NUMBER TIME TRANSLATION OPERATOR u

1) Introduction

In Chapter IV, the pair creation amplitude was calculated using a perturbation expansion for the time translation operator u. In this Chapter much will be determined about u without resorting to perturbation theory. In Chapter VI, this information will be used to attempt to calculate the pair creation amplitude with an exact operator u.

The Differential Equation

The quantized particle field at time t may be related to that at time t' by (II-2.3):

$$\psi(\underset{\sim}{x},t) = u(t,t')\psi(\underset{\sim}{x},t') \ . \tag{1.1}$$

Here u(t,t') is a unitary operator on $[\mathcal{L}^2(R^3)]^4$ with properties (II-2.2):

$$i\,\frac{\partial}{\partial t}\,u(t,t') = h(t)u(t,t') \ , \tag{1.2}$$

$$u(t,t) = 1 \ , \tag{1.3}$$

and

$$u(t_3,t_2)u(t_2,t_1) = u(t_3,t_1) \ . \tag{1.4}$$

In the above formulae, the time dependence of u is shown

explicitly and its spacial dependence is suppressed. In situations where the times of interest are fixed, the spacial dependence of u may be shown explicitly:

$$\psi(\underset{\sim}{x},t) = u(\underset{\sim}{x})\psi(\underset{\sim}{x},t') \quad . \tag{1.5}$$

Remember, u is an operator - by (1.2), it will incorporate $\underset{\sim}{\partial}$ (through h_o) and the values of $A(\underset{\sim}{x},\tau)$ for $t' < \tau < t$.

If u were just a c-number function, the solution of (1.2) subject to (1.3) would be:

$$u(t,t') = \exp(-i \int_{t'}^{t} h(\tau)d\tau) \quad . \tag{1.6}$$

However, the operator nature of h(t) makes the problem more complicated. Since,

$$[h(t_1),h(t_2)] \neq 0$$

$$\text{unless } A(\underset{\sim}{x},t_1) = A(\underset{\sim}{x},t_2) \tag{1.7}$$

differentiation of (1.6) by t does not yield the c-number result; that is

$$-i \frac{\partial}{\partial t} \exp(-i \int_{t'}^{t} h(\tau)d\tau) \neq h(t)\exp(-i \int_{t'}^{t} h(\tau)d\tau),$$

$$\text{unless } A(\underset{\sim}{x},\tau) = A(\underset{\sim}{x},t) \ , \quad \forall \quad t' \leq \tau \leq t \ . \tag{1.8}$$

A simple solution to (1.2) is not obtained for a time dependent field.

If the external field is a smooth function of time, an approximate solution to (1.2) may be found. Let:

$$\varepsilon = (t_2 - t_1)/N \quad . \tag{1.9}$$

Then, using (1.4),

$$u(t_2,t_1) \cong \prod_{j=1}^{N} \exp(-i\varepsilon h(t_1 + j\varepsilon)) \tag{1.10}$$

where the j = 1 term is rightmost in the product. Taking the limit $N \to \infty$ yields an exact $u(t_2,t_1)$. Notice that this limit of (1.10) is not (1.6) since, when operators are involved, the exponential of a sum is not the product of exponentials. While the limit of (1.10) may be an exact solution to (1.2) it is not a very useful expression with which to carry out computations such as (III-6.14).

Some Soluble Problems

Consider firstly a free field. By (IV-1.4), u is known:

$$u(t,t') = e^{-i(t-t')h_o} \quad . \tag{1.11}$$

The time translation operator for a time independent external field was found in (II-1.2):

$$u(t,t') = e^{-i(t-t')h} \quad . \tag{1.12}$$

Using (1.4), it is possible to find u for an external field with step function time dependence. Suppose the external field is:

$$A(\underset{\sim}{x},t) = \theta(t_o-t)A(\underset{\sim}{x}) + \theta(t-t_o)A'(\underset{\sim}{x}) \ . \tag{1.13}$$

Define:

$$h = h_o - e\gamma^o\gamma.A(\underset{\sim}{x}) \ ,$$

and $$h' = h_o - e\gamma^o\gamma\cdot A'(\underset{\sim}{x}) \ . \tag{1.14}$$

The time translation operator is:

$$u(t,t') = e^{-ih(t-t')} \qquad \text{if } t<t_o,\ t'<t_o$$

$$u(t,t') = e^{-ih'(t-t_o)}e^{-ih(t_o-t')} \qquad \text{if } t>t_o,\ t'<t_o$$

and

$$u(t,t') = e^{-ih'(t-t')} \qquad \text{if } t>t_o,\ t'>t_o. \tag{1.15}$$

A similar application of (1.4) may be used to obtain u for an external field which has many steps.

Another external field for which the time translation operator may be computed was found by G. Labonté [1973]:

$$A(\underset{\sim}{x},t) = \delta(t)A(\underset{\sim}{x}) \ ,$$

where

$$A(\underset{\sim}{x}) = (A_o(\underset{\sim}{x}),0,0,0) \ . \tag{1.16}$$

Here:

$$u(t,t') = e^{-ih_o(t-t')} \qquad \text{if } t<0,\ t'<0 \ \text{ or } \ t>0,\ t'>0,$$

and

$$u(t,t') = e^{-ih_o t}M(\underset{\sim}{x})e^{ih_o t'} \qquad \text{if } t>0 \ \text{ and } \ t'<0 \ . \tag{1.17}$$

The operator $M(\underset{\sim}{x})$ is defined by:

$$\psi^{out}(0,\underset{\sim}{x}) = M(\underset{\sim}{x})\psi^{in}(0,\underset{\sim}{x}) \tag{1.18}$$

and is:

$$M(\underset{\sim}{x}) = 1 + \frac{ieA_o(\underset{\sim}{x})}{1 - i\,\frac{e}{2}\,A_o(\underset{\sim}{x})} = 1 + F(\underset{\sim}{x}) \ . \tag{1.19}$$

All the Bogoliubov transformations are unitarily implementable if

$$\int d^3q\ q^{3/2}\ |\tilde{F}(q)|^2 < \infty \ . \tag{1.20}$$

In this case, $A_o(\underset{\sim}{x})$ could behave badly as a function of $\underset{\sim}{x}$ and (1.18) would still hold.

<u>Decomposition of u</u>

Since u is an operator on $[\mathcal{L}^2(R^3)]^4$, it may be decomposed according to its matrix structure:

$$u = \sum_{\alpha=1}^{16} u_\alpha \Gamma_\alpha \tag{1.21}$$

where u_α is an operator on $\mathcal{L}^2(R^3)$ and Γ_α is one of the sixteen Dirac matrices.

Consider two external fields $A(x)$ and $A'(x)$. By (1.2) the time translation operator u for the first problem will depend on $A(x)$ in exactly the same way as the operator u' for the second problem will depend on $A'(x)$. From (1.21), it is seen that the dependence of u_α on A is also the same as that of u'_α on A'. These facts will be useful in trying to determine the exact form of the

operators u_α.

Symmetries

The Dirac equation (I-1.1) is "invariant" under many kinds of transformations. In the context of an external field problem, invariance has the following meaning: if all the quantities in the Dirac equation are transformed in a certain way, then the transformed quantities also satisfy a Dirac equation. That is:

$$x \to x' \quad ,$$

$$\partial \to \partial' \quad ,$$

$$\psi(x) \to \psi'(x') \quad ,$$

$$A(x) \to A'(x') \quad ,$$

and

$$(-i\gamma\cdot\partial' + m)\psi'(x') = e\gamma\cdot A'(x')\psi'(x') \quad . \tag{1.22}$$

This transformation (1.22) is best pictured as a change of observation frame. The unprimed variables and fields refer to one frame; the primed variables and fields correspond to those in another. For example, in charge conjugation the primed observer sees charges of $-e$ whenever the unprimed observer sees charges of $+e$ - the voltmeters in the two frames are wired oppositely. Unless the external field has some special symmetry, the external field observed in the primed frame is different from that in the unprimed frame. Both observers may use the external fields that they themselves observe

and work out Bogoliubov transformations, creation amplitudes and so on. The same physical principles hold in both frames.

Some of the transformations which leave the Dirac equation invariant include charge conjugation, time reversal, parity change, Lorentz transformations and gauge transformations.

The quantized particle field ψ' describes the electron-positron field in a transformed frame. It is meaningful to ask how ψ' depends on the new time variable t'. ψ' may be decomposed in a manner similar to ψ according to an equation like (I-2.1) with all quantities primed. In partciular, a c-number time translation operator u' exists such that:

$$\psi'(\underset{\sim}{x}',t_2') = u'(t_2',t_1')\psi'(\underset{\sim}{x}',t_1') \qquad (1.23)$$

For some types of transformations, the relations (1.22) together with the definition (1.23) allow the relationship between u and u' to be deduced. Since u depends on the unprimed quantities in exactly the same way as u' depends on the primed ones, knowing the behaviour of u under a symmetry transformation may help to determine the structure of u.

In sections 3) and 4) the transformation properties of u and the u_α will be investigated and the results used to gain more information about the form of these operators.

2) The Time Translation Properties of u_α

The Differential Equation

According to (1.21), the time translation operator u is decomposed:

$$u = u_I I + \underset{\sim}{u}\cdot\underset{\sim}{\gamma} + u_o\gamma_o + \underset{\sim}{u}_o\cdot\gamma_o\underset{\sim}{\gamma} + u_{o5}\gamma_o\gamma_5 + \underset{\sim}{u}_\sigma\cdot\underset{\sim}{\sigma} + u_5\gamma_5 + \underset{\sim}{u}_{o\sigma}\cdot\gamma_o\underset{\sim}{\sigma}. \tag{2.1}$$

Substitution of this expansion into the differential equation (1.2) for u yields a system of differential equations for the u_α:

$$i(\frac{\partial}{\partial t} - ieA_o)u_I = mu_o + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\cdot\underset{\sim}{u}_o \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)\underset{\sim}{u} = m\underset{\sim}{u}_o - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_o - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\times\underset{\sim}{u}_{o\sigma} \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)u_o = mu_I - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\cdot\underset{\sim}{u} \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)\underset{\sim}{u}_o = m\underset{\sim}{u} + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_I + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\times\underset{\sim}{u}_5 \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)u_{o5} = mu_5 - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\cdot\underset{\sim}{u}_{o\sigma} \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)\underset{\sim}{u}_\sigma = m\underset{\sim}{u}_{o\sigma} + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_5 - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\times\underset{\sim}{u}_o \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)u_5 = mu_{o5} + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\cdot\underset{\sim}{u}_\sigma \ ,$$

$$i(\frac{\partial}{\partial t} - ieA_o)\underset{\sim}{u}_{o\sigma} = m\underset{\sim}{u}_\sigma - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_{o5} + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\times\underset{\sim}{u} \ . \tag{2.2}$$

This system of differential equations is to be solved

subject to the initial condition:

$$u_I(t_o,t_o) = 1 \ ,$$

$$u_\alpha(t_o,t_o) = 0 \qquad \text{if} \quad \alpha \neq I \ . \tag{2.3}$$

Just as the operator nature of u resulted in difficulties in (1.6)-(1.8), the operator nature of the u_α is a problem here. The system (2.2) has the appearance of a system of first order linear differential equations in the variable t. If the "coefficient" $(\underset{\sim}{\partial} - ie\underset{\sim}{A})$ were simply a function of time, standard techniques could be applied to integrate the system. However, since

$$i(\underset{\sim}{\partial} - ie\underset{\sim}{A}) \times i(\underset{\sim}{\partial} - ie\underset{\sim}{A}) = ie\underset{\sim}{\nabla} \times \underset{\sim}{A} = ie\underset{\sim}{B} \ ,$$

and

$$[i(\frac{\partial}{\partial t} - ieA_o), i(\underset{\sim}{\partial} - ie\underset{\sim}{A})] = ie(\frac{\partial}{\partial t}\underset{\sim}{A} - \underset{\sim}{\nabla}A_o) = ie\underset{\sim}{E} \tag{2.4}$$

usual methods are inapplicable. This system will be examined in a few special cases shortly.

The groups of operators $(u_I,\underset{\sim}{u})$, $(u_o,\underset{\sim}{u}_o)$, $(u_{o5},\underset{\sim}{u}_o)$ and $(u_5,\underset{\sim}{u}_{o\sigma})$ play similar roles in this system of differential equations. In fact (2.2) remains unchanged under the replacement:

$$\begin{aligned}
(u_I,\underset{\sim}{u}) &\to (u_o,-\underset{\sim}{u}_o) \ , \\
(u_o,\underset{\sim}{u}_o) &\to (u_I,-\underset{\sim}{u}) \ , \\
(u_{o5},\underset{\sim}{u}_\sigma) &\to (-u_5,\underset{\sim}{u}_{o\sigma}) \ , \\
(u_5,\underset{\sim}{u}_{o\sigma}) &\to (-u_{o5},\underset{\sim}{u}_\sigma) \ ,
\end{aligned} \tag{2.5}$$

and under the replacement:

$$(u_I,\underset{\sim}{u}) \rightarrow (u_{o5},-\underset{\sim}{u}_\sigma) \quad ,$$

$$(u_o,u_o) \rightarrow (u_5,-u_{o\sigma}) \quad ,$$

$$(u_{05},\underset{\sim}{u}_\sigma) \rightarrow (u_I,-\underset{\sim}{u}) \quad ,$$

$$(u_5,\underset{\sim}{u}_{o\sigma}) \rightarrow (u_o,-\underset{\sim}{u}_o) \quad . \tag{2.6}$$

This is just a reflection of the algebraic properties of the Dirac matrices.

The solution for the free field has been found earlier:

$$u(t,t') = e^{-i(t-t')h_o} \quad . \tag{2.7}$$

Since

$$h_o = \gamma_o(i\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial} + m) \quad ,$$

$$h_o^{2n} = (-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^n \quad ,$$

and

$$h_o^{2n+1} = (-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^n \ \gamma_o(i\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial} + m) \quad , \tag{2.8}$$

the decomposition of u(t,t') may easily be obtained:

$$u(t,t') = I \ \cos[(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}(t-t')] - \frac{i\gamma^o(i\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial} + m)}{(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}} \times$$

$$\times \ \sin[(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}(t-t')] \quad . \tag{2.9}$$

In other words:

$$u_I(t,t') = \cos[(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}(t-t')] \quad ,$$

$$u_o(t,t') = \frac{-im}{(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}} \ \sin[(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}(t-t')] \quad ,$$

and

$$u_o = \frac{\underset{\sim}{\partial}}{(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}} \sin[(-\underset{\sim}{\partial}\cdot\underset{\sim}{\partial} + m^2)^{\frac{1}{2}}(t-t')] . \qquad (2.10)$$

These operators are seen to satisfy (2.2) with $A \equiv 0$, as required.

Perhaps there are other external fields for which so many of the u_α are zero. Assume $\underset{\sim}{u} = \underset{\sim}{u}_\sigma = \underset{\sim}{u}_{o\sigma} = 0$ and $u_5 = u_{o5} = 0$ - these u_α were found to be zero for the free field. What kind of field may produce such an operator u? The system (2.2) becomes:

$$i(\frac{\partial}{\partial t} - ieA_o)u_I = mu_o + i(\underset{\sim}{\partial} - ie\underset{\sim}{A})\cdot\underset{\sim}{u}_o \ , \qquad (2.11)$$

$$i(\frac{\partial}{\partial t} - ieA_o)u_o = mu_I \ , \qquad (2.12)$$

$$0 = m\underset{\sim}{u}_o - i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_o \ , \qquad (2.13)$$

$$i(\frac{\partial}{\partial t} - ieA_o)\underset{\sim}{u}_o = i(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_I \ , \qquad (2.14)$$

$$0 = -i(\underset{\sim}{\partial} - ie\underset{\sim}{A}) \times \underset{\sim}{u}_o \ . \qquad (2.15)$$

The operator u_o may be used to determine u_I and $\underset{\sim}{u}_o$ according to (2.12) and (2.13):

$$u_I = \frac{i}{m}(\frac{\partial}{\partial t} - ieA_o)u_o$$

$$\underset{\sim}{u}_o = \frac{i}{m}(\underset{\sim}{\partial} - ie\underset{\sim}{A})u_o \ . \qquad (2.16)$$

Combining (2.16) with (2.14) and with (2.15) gives:

$$(\frac{\partial}{\partial t} \underset{\sim}{A} - \underset{\sim}{\nabla} A_o) u_o = 0$$

$$(\underset{\sim}{\nabla} \times \underset{\sim}{A}) u_o = 0 \quad . \tag{2.17}$$

Since u_o must be non-zero to give any solution at all, (2.17) means:

$$\underset{\sim}{E} = \underset{\sim}{B} = 0 \ . \tag{2.18}$$

So it is seen that a time translation operator u which has a matrix structure with $\underset{\sim}{u} = \underset{\sim}{u}_\sigma = \underset{\sim}{u}_{o\sigma} = 0$ and $u_5 = u_{o5} = 0$ is simply the time translation operator for the free field problem.

Even in a time independent external field, it is clear that generally no u_α is zero. For $A_o \equiv 0$,

$$u(t_2,t_1) = e^{i(t_2-t_1)h} = e^{-i(t_2-t_1)\gamma^o(i\underset{\sim}{\gamma}\cdot(\underset{\sim}{\partial}-ie\underset{\sim}{A})+m)} \ . \tag{2.19}$$

In expanding (2.19) in a power series in h, a term in $\underset{\sim}{\sigma}$ is found in second order. This follows since

$$(\underset{\sim}{\partial} - ie\underset{\sim}{A}) \times (\underset{\sim}{\partial} - ie\underset{\sim}{A}) = -ie\underset{\sim}{B} \not\equiv 0 \ . \tag{2.20}$$

In higher orders, combinations of γ^o, $\gamma^o\underset{\sim}{\gamma}$ and $\underset{\sim}{\sigma}$ occur thereby producing all possible Γ_α. Only fortuitous cancellations would result in one u_α being zero. If $\underset{\sim}{A} \equiv 0$,

$$u(t_2,t_1) = e^{-i(t_2-t_1)[\gamma^o(i\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial}+m)-eA_o]} \ . \tag{2.21}$$

The term $(-ie\gamma^o\underset{\sim}{\gamma}\cdot\underset{\sim}{\nabla}A_o - 2ieA_o\gamma^o\underset{\sim}{\gamma}\cdot\underset{\sim}{\partial})$ arising in second order produces $\underset{\sim}{\sigma}$ terms in third order and again all the u_α are necessary.

A Second Order System

The system (2.2) may be changed into a second order system by application of $i(\frac{\partial}{\partial t} - ieA_o)$ to all the equations. Using (2.4), it is found that:

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]u_I = -ie\underset{\sim}{B}\cdot\underset{\sim}{u}_\sigma - ie\underset{\sim}{E}\cdot\underset{\sim}{u}_o$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]\underset{\sim}{u} = -ie\underset{\sim}{B}u_{o5} + ie\underset{\sim}{E}u_o + ie\underset{\sim}{E} \times \underset{\sim}{u}_{o\sigma}$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]u_o = -ie\underset{\sim}{B}\cdot\underset{\sim}{u}_{o\sigma} + ie\underset{\sim}{E}\cdot\underset{\sim}{u}$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]\underset{\sim}{u}_o = -ie\underset{\sim}{B}u_5 - ie\underset{\sim}{E}u_5 - ie\underset{\sim}{E} \times \underset{\sim}{u}_\sigma$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]u_{o5} = ie\underset{\sim}{B}\cdot\underset{\sim}{u} + ie\underset{\sim}{E}\cdot\underset{\sim}{u}_{o\sigma}$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]\underset{\sim}{u}_\sigma = ie\underset{\sim}{B}u_I - ie\underset{\sim}{E}u_5 + ie\underset{\sim}{E} \times \underset{\sim}{u}_o$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]u_5 = ie\underset{\sim}{B}\cdot\underset{\sim}{u}_o - ie\underset{\sim}{E}\cdot\underset{\sim}{u}_\sigma$$

$$[(\frac{\partial}{\partial t} - ieA_o)^2 - (\underset{\sim}{\partial} - ie\underset{\sim}{A})^2 + m^2]\underset{\sim}{u}_{o\sigma} = ie\underset{\sim}{B}u_o + ie\underset{\sim}{E}u_{o5} - ie\underset{\sim}{E} \times \underset{\sim}{u}. \tag{2.22}$$

Unitarity

The c-number time translation operator u is unitary and satisfies the composition requirement (1.4). From these properties, it is obtained that:

$$u^\dagger(t_2,t_1) = u(t_1,t_2) \ . \tag{2.23}$$

For the u_α, (2.20) is interpreted:

$$u_I^\dagger(t_2,t_1) = u_I(t_1,t_2) \quad ,$$

$$\underset{\sim}{u}^\dagger(t_2,t_1) = -\underset{\sim}{u}(t_1,t_2) \quad ,$$

$$u_o^\dagger(t_2,t_1) = u_o(t_1,t_2) \quad ,$$

$$\underset{\sim}{u}_o^\dagger(t_2,t_1) = \underset{\sim}{u}_o(t_1,t_2) \quad ,$$

$$u_{o5}^\dagger(t_2,t_1) = u_{o5}(t_1,t_2) \quad ,$$

$$\underset{\sim}{u}_\sigma^\dagger(t_2,t_1) = -\underset{\sim}{u}_\sigma(t_1,t_2) \quad ,$$

$$u_5^\dagger(t_2,t_1) = -u_5(t_1,t_2) \quad ,$$

$$\underset{\sim}{u}_{o\sigma}^\dagger(t_2,t_1) = -\underset{\sim}{u}_{o\sigma}(t_1,t_2) . \qquad (2.24)$$

<u>The composition Relation</u>

The relation (1.4) has not yet been expressed in terms of the decomposition of u:

$$u(t_3,t_1) = \sum_{\alpha,\beta} u_\alpha(t_3,t_2)u_\beta(t_2,t_1)\Gamma_\alpha\Gamma_\beta \quad . \qquad (2.25)$$

This matrix multiplication could be carried out and the coefficient of Γ_γ found, thereby determining $u_\gamma(t_3,t_1)$. The sixteen long relations that result can give no further information about the operators u_α beyond (2.2) and (2.24) since (1.4) is derived from (1.2) and (1.3). The complicated nature of these relations makes them even less transparent than the relations (2.2) and (2.24) from which they

may be obtained. For this reason, the consequences of (2.25) are of little interest in understanding the structure of the u_α but could be of use in some explicit calculation.

3) The Symmetry Properties of u_α

Charge Conjugation

The Dirac equation is invariant under charge conjugation. This transformation has the following effect:

$$
\begin{aligned}
&x \to x' = x \ , \\
&\psi(x) \to \psi'(x') = \gamma_2 \psi^\times(x) \ , \\
&A(x) \to A'(x') = -A(x) \ , \\
&b_\beta(t) \to b'_\beta(t') = d_\beta(t) \ , \\
&d_\beta(t) \to d'_\beta(t') = b_\beta(t) \ , \\
&|0(t)\rangle \to |0(t')\rangle' = |0(t)\rangle \ .
\end{aligned}
\qquad (3.1)
$$

The operation $\times$ is Hermitian conjugation for q-number operators and complex conjugation for c-number quantities. Charge conjugation interchanges the roles of electron and positron.

The c-number time translation operator u^c for the transformed particle field relates $\psi'(\underset{\sim}{x},t_1)$ and $\psi'(\underset{\sim}{x},t_2)$:

$$\psi'(\underset{\sim}{x},t_2) = u^c(t_2,t_1)\psi'(\underset{\sim}{x},t_1) \quad . \tag{3.2}$$

Using (3.1) in (3.2) it is found that:

$$\psi(\underset{\sim}{x},t_2) = -\gamma_2[u^c(t_2,t_1)]^*\gamma_2\psi(\underset{\sim}{x},t_1) \quad , \tag{3.3}$$

and so:

$$u(t_2,t_1) = -\gamma_2[u^c(t_2,t_1)]^*\gamma_2 \quad . \tag{3.4}$$

This relation is interpreted in terms of the operators u_α and u^c_α in section 4).

<u>Parity</u>

The following transformation changes the parity of the system:

$$\underset{\sim}{x} \rightarrow \underset{\sim}{x}' = \underset{\sim}{x} \quad ,$$

$$t \rightarrow t' = t \quad ,$$

$$\psi(x) \rightarrow \psi'(x') = \gamma_o\psi(x) \quad ,$$

$$\underset{\sim}{A}(x) \rightarrow \underset{\sim}{A}'(x') = -\underset{\sim}{A}(x) \quad ,$$

$$A_o(x) \rightarrow A'_o(x') = A_o(x) \quad ,$$

$$b_\beta(t) \rightarrow b'_\beta(t') = b_{P\beta}(t) \quad ,$$

$$d_\beta(t) \rightarrow d'_\beta(t') = d_{P\beta}(t) \quad ,$$

$$|0(t)> \rightarrow |0(t')>' = |0(t)> \quad . \tag{3.5}$$

The quantum numbers $P\beta$ are defined to have the opposite momentum and the same spin projection as β.

The time translation operator u^P satisfies:

$$\psi'(\underset{\sim}{x}',t_2) = u^P(t_2,t_1)\psi'(\underset{\sim}{x}',t_1) \ . \qquad (3.6)$$

This leads to:

$$u(t_2,t_1) = \gamma_o u^P(t_2,t_1)\gamma_o \ . \qquad (3.7)$$

The results for the operators u_α and u_α^P are found in section 4).

Time Reversal

The time reversal transformation is also of interest:

$$t \to t' = -t \quad ,$$

$$\underset{\sim}{x} \to \underset{\sim}{x}' = \underset{\sim}{x} \quad ,$$

$$\psi(x) \to \psi'(x') = \gamma_1\gamma_3\psi^*(x) \quad ,$$

$$\underset{\sim}{A}(x) \to \underset{\sim}{A}'(x') = -\underset{\sim}{A}(x) \quad ,$$

$$A_o(x) \to A_o'(x') = A_o(x) \quad ,$$

$$b_\beta(t) \to b_\beta'(t') = b_{T\beta}^*(t) \quad ,$$

$$d_\beta(t) \to d_\beta'(t') = d_{T\beta}^*(t) \quad ,$$

$$|0(t)\rangle \to |0(t')\rangle' = |0(t)\rangle^* . \qquad (3.8)$$

The quantum numbers $T\beta$ are defined to have opposite momentum and spin projection to β.

The time translation operator u^T satisfies:

$$\psi'(\underset{\sim}{x},t_2) = u^T(t_2,t_1)\psi'(\underset{\sim}{x},t_1) \quad , \tag{3.9}$$

and so

$$\psi(\underset{\sim}{x},-t_2) = \gamma_3\gamma_1[u^T(t_2,t_1)]^*\gamma_1\gamma_3\psi(\underset{\sim}{x},-t_1) \quad . \tag{3.10}$$

The relationship between u and u^T is therefore seen to be:

$$u(t_2,t_1) = \gamma_3\gamma_1[u^T(-t_2,-t_1)]^*\gamma_1\gamma_3 \quad . \tag{3.11}$$

The expansion of (3.11) in the operators u_α and u_α^T is found in section 4).

Lorentz Transformations

The Dirac equation transforms under Lorentz transformation according to:

$$x^\mu \rightarrow x'^\mu = a^\mu{}_\nu \, x^\nu \quad ,$$

$$\psi(x) \rightarrow \psi'(x') = S\psi(x) \quad ,$$

$$A^\mu(x) \rightarrow A'^\mu(x') = a^\mu{}_\nu \, A^\nu(x),$$

where $$S\gamma_\mu S^{-1} a_\nu{}^\mu = \gamma_\nu \quad . \tag{3.12}$$

The external field A'(x') is the field as seen in the new moving frame. This external field problem may be solved in the same manner as the old external field problem was in the old frame - auxiliary fields may be found for ψ' for each value of t'; a vacuum $|0(t')\rangle'$ with creation operators $b_\beta'^\dagger(t')$ and $d_\beta'^\dagger(t')$ make sense; Bogoliubov

transformations and c-number time translation operators play analogous roles in the solution of the transformed problem.

A Lorentz transformation mixes time and space so that simultaneity in one frame does not correspond to simultaneity in another frame. For this reason it is impossible to transform between $|0(t)\rangle$ and $|0(t')\rangle'$, $b_\beta(t)$ and $b'_\beta(t')$, $u(t_2,t_1)$ and $u'(t'_2,t'_1)$ and many other quantities just dependent on time. The vacuum state $|0(t)\rangle$ describes the configuration of the system at the time t in which no particles are present anywhere in space. The operator $b^\dagger_\beta(t)$ creates an electron of type β somewhere in space at the time t - particles localized in space may be defined using superpositions of creation operators. The operator $u(t_2,t_1)$ relates the values of ψ at one spacial point between the two times t_1 and t_2. Time and space are used very differently in all these quantities. For a moving frame a surface, in four dimensional space time, of constant time is quite different from a surface of constant time for the original frame - it contains points corresponding to many times in the original frame. If an observer in a moving frame says there are no particles present at a given time, an observer in the original frame knows that at a certain collection of positions and times he would see no particles - a vacuum state to one observer may be interpreted

by another observer but the interpretation is not in terms of vacuum and particle states. A similar situation occurs with u and u' as shown in the next calculation.

An attempt will be made to find the relationship between u and u'.

$$\psi'(\underset{\sim}{x},t_2) = u'(t_2,t_1)\psi'(\underset{\sim}{x},t_1) \; . \qquad (3.13)$$

So

$$S\psi(L^{-1}(\underset{\sim}{x},t_2)) = u'(t_2,t_1)S\psi(L^{-1}(\underset{\sim}{x},t_1)) \qquad (3.14)$$

and

$$\psi(L^{-1}(\underset{\sim}{x},t_2)) = S^{-1}u'(t_2,t_1)S\psi(L^{-1}(\underset{\sim}{x},t_1)) \; . \qquad (3.15)$$

The operator $S^{-1}u'(t_2,t_1)S$ relates ψ at one place and one time to ψ at another place and another time - this operator isn't even a time translation operator.

Those Lorentz transformations which involve a boost are seen to be of no use in investigating the time translation operator u. The purely spacial transformations of rotations and translations may be of assistance.

<u>Translations</u>

The Dirac equation is invariant under spatial translations:

$$\underset{\sim}{x} \to \underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{a} \;, \qquad \underset{\sim}{a} \in R^3$$

$$t \to t' = t \qquad ,$$

$$A(x) \to A'(x') = A(x) \;,$$

$$\psi(x) \to \psi'(x') = \psi(x) \;. \qquad (3.16)$$

The relationship between u and u' is found using (3.16) and

$$\psi'(\underset{\sim}{x}',t_2) = u'(t_2,t_1)\psi'(\underset{\sim}{x}',t_1) \;, \qquad (3.17)$$

and is seen to be:

$$u(t_2,t_1) = u'(t_2,t_1) \;. \qquad (3.18)$$

Earlier arguments have shown that u' depends on A' in the same way as u depends on A. If one had an algorithm for appropriately integrating and expontentiating A to get u, this algorithm could be used to get u' from A'. For translations, $A(x) = A'(x')$ and $u(\underset{\sim}{x}) = u'(\underset{\sim}{x}')$; translation symmetries, therefore, don't help unravel this algorithm.

Rotations

Another transformation under which the Dirac equation is invariant is that of spacial rotations:

$$\underset{\sim}{x} \to x' = R\underset{\sim}{x} \quad ,$$

$$t \to t' = t \quad ,$$

$$\underset{\sim}{A}(x) \to \underset{\sim}{A}'(x') = R\underset{\sim}{A}(x) \;,$$

$$A_o(x) \to A_o'(x') = A_o(x) \ ,$$

$$\psi(x) \to \psi'(x') = S\psi(x) \ , \qquad (3.19)$$

with

$$S\gamma_o S^{-1} = \gamma_o \ ,$$

and

$$S\underset{\sim}{\gamma}S^{-1} = R^{-1}\underset{\sim}{\gamma} \ . \qquad (3.20)$$

The relationship between u and u^R is sought:

$$\psi'(\underset{\sim}{x}',t_2) = u^R(t_2,t_1)\psi'(\underset{\sim}{x}',t_1) \ , \qquad (3.21)$$

and so

$$u(t_2,t_1) = S^{-1}u^R(t_2,t_1)S \ . \qquad (3.22)$$

Using (3.20), it is found that:

$$\begin{aligned}
u_I &= u_I^R \\
\underset{\sim}{u} &= R^{-1}\underset{\sim}{u}^R \\
u_o &= u_o^R \\
\underset{\sim}{u}_o &= R^{-1}\underset{\sim}{u}_o^R \\
u_{o5} &= u_{o5}^R \\
\underset{\sim}{u}_\sigma &= R^{-1}\underset{\sim}{u}_\sigma^R \\
u_5 &= u_5^R \\
\underset{\sim}{u}_{o\sigma} &= R^{-1}\underset{\sim}{u}_{o\sigma} \ .
\end{aligned} \qquad (3.23)$$

The u_α are constructed from the quantities $\underset{\sim}{\partial}$ and $\underset{\sim}{A}$ in such a way that u_I, u_o, u_{o5} and u_5 are rotationally invariant and $\underset{\sim}{u}$, $\underset{\sim}{u}_o$, $\underset{\sim}{u}_\sigma$ and $\underset{\sim}{u}_{o\sigma}$ rotate like vectors as

$\underset{\sim}{\partial}$ and $\underset{\sim}{A}$ rotate.

Gauge Transformations

Another transformation under which the Dirac equation is invariant is a gauge transformation:

$$x \to x' = x \ ,$$

$$\psi(x) \to \psi'(x) = \exp(ie\Lambda(x))\psi(x) \ ,$$

$$A_\mu(x) \to A'_\mu(x') = A_\mu(x) + \frac{\partial\Lambda(x)}{\partial x^\mu} \ . \qquad (3.24)$$

The function $\Lambda(x)$ is arbitrary or may be required to satisfy $\Box\Lambda(x) = 0$, if the Lorentz gauge is desired for $A(x)$.

How does $u(x)$ behave under a gauge transformation?

$$\psi'(\underset{\sim}{x},t_2) = u^G(t_2,t_1)\psi'(\underset{\sim}{x},t_1) \ . \qquad (3.25)$$

Hence:

$$u(t_2,t_1) = \exp(-ie\Lambda(\underset{\sim}{x},t_2))u^G(t_2,t_1)\exp(ie\Lambda(\underset{\sim}{x},t_1)) \ , \qquad (3.26)$$

and

$$u_\alpha(t_2,t_1) = \exp(-ie\Lambda(\underset{\sim}{x},t_2))u^G_\alpha(t_2,t_1)\exp(ie\Lambda(\underset{\sim}{x},t_1)) \ . \qquad (3.27)$$

Bogoliubov Identities

The time translation operator u also appears in the relations (III-1.4) and (III-1.5), through the definition of the M_i matrices. Perhaps some useful property of u may be discovered here.

Rewrite (III-1.4) using (IV-6.6):

$$\sum_{\alpha}(f^{o}_{+\gamma},uf^{o}_{+\alpha})(f^{o}_{+\alpha},u^{\dagger}f^{o}_{+\gamma'}) + (f^{o}_{+\gamma},uf^{o}_{-\alpha})(f^{o}_{-\alpha},u^{\dagger}f^{o}_{+\gamma'}) = \delta_{\gamma\gamma'} \quad . \tag{3.28}$$

Since $\{f^{o}_{+\alpha},f^{o}_{-\alpha}\}$ is a complete set, this becomes:

$$(f^{o}_{+\gamma},uu^{\dagger}f^{o}_{+\gamma'}) = \delta_{\gamma\gamma'} \quad . \tag{3.29}$$

The other Bogoliubov identities are:

$$(f^{o}_{-\lambda},uu^{\dagger}f^{o}_{-\lambda'}) = \delta_{\lambda\lambda'} \quad ,$$

$$(f^{o}_{+\gamma},uu^{\dagger}f^{o}_{-\lambda}) = 0 \quad ,$$

$$(f^{o}_{+\alpha},u^{\dagger}uf^{o}_{+\alpha'}) = \delta_{\alpha\alpha'} \quad ,$$

$$(f^{o}_{-\beta},u^{\dagger}uf^{o}_{-\beta'}) = \delta_{\beta\beta'} \quad ,$$

and

$$(f^{o}_{+\alpha},u^{\dagger}uf^{o}_{-\beta}) = 0 \quad . \tag{3.30}$$

The relations (2.29) and (3.30) just reconfirm that u is unitary.

4) Summary of Properties of u_α

In sections 2) and 3) many properties of the operators u_α have been found. The operators u_I, u_o, u_{o5} and u_5 are scalars and $\underset{\sim}{u}$, $\underset{\sim}{u}_o$, $\underset{\sim}{u}_\sigma$ and $\underset{\sim}{u}_{o\sigma}$ are vectors under spatial rotations. The unitary nature of u and the C.P.T. invariance of the Dirac equation give:

$$u_\alpha(t_2,t_1) = \varepsilon_\alpha^U[u_\alpha(t_1,t_2)]^\dagger , \tag{4.1a}$$

$$= \varepsilon_\alpha^C[u_\alpha^C(t_2,t_1)]^* , \tag{4.1b}$$

$$= \varepsilon_\alpha^P[u_\alpha^P(t_2,t_1)] , \tag{4.1c}$$

$$= \varepsilon_\alpha^T[u_\alpha^T(-t_2,-t_1)]^* , \tag{4.1d}$$

where Table 1 gives ε_α^U, ε_α^C, ε_α^P, ε_α^T.

Γ_α	ε_α^U	ε_α^C	ε_α^P	ε_α^T
I	+	+	+	+
$\underset{\sim}{\gamma}$	-	-	-	-
γ_o	+	-	+	+
$\gamma_o\underset{\sim}{\gamma}$	+	+	-	-
$\gamma_o\gamma_5$	+	-	-	-
$\underset{\sim}{\sigma}$	-	+	+	+
γ_5	-	+	-	-
$\gamma_o\underset{\sim}{\sigma}$	-	-	+	+

<u>Table 1</u> Constants appearing in symmetry relations for u_α.

As operators on $\mathcal{L}^2(R^3)$, the u_α may be constructed out of the complete set of operators $\{\partial_i, X_i \mid i = 1,2,3\}$. The operation X_i is multiplication by the variable x_i. In other words, each u_α may be thought of as a power series in $\{\partial_i, X_i\}$ or as a power series in $\{\partial_i\}$ with complex valued functions of $\underset{\sim}{x}$ as coefficients. The differential

equations (2.2) indicate that only the functions $A(\underset{\sim}{x},t)$ and $A_o(\underset{\sim}{x},t)$, with $t_1 \leq t \leq t_2$, contribute and in fact $\underset{\sim}{A}(\underset{\sim}{x},t)$ only occurs in the combination $(\underset{\sim}{\partial} - ie\underset{\sim}{A}(\underset{\sim}{x},t))$; some sort of time integration similar to the suggestion of (1.6) is needed.

To show the dependence of u_α on the external field and $\underset{\sim}{\partial}$ explicitly, write:

$$u_\alpha(t_2,t_1) = u_\alpha((\underset{\sim}{\partial} - ie\underset{\sim}{A}(x)),A_o(x);\ t \in (t_1,t_2)). \qquad (4.2)$$

The transformation properties of $\underset{\sim}{\partial}$, A and t under the C.P.T. symmetry transformations give:

$$u_\alpha^C(t_2,t_1) = u_\alpha((\underset{\sim}{\partial} + ie\underset{\sim}{A}(x)),-A_o(x);\ t \in (t_1,t_2)),$$

$$u_\alpha^P(t_2,t_1) = u_\alpha(-(\underset{\sim}{\partial} - ie\underset{\sim}{A}(\underset{\sim}{x})),\ A_o(x);\ t \in (t_1,t_2)),$$

$$u_\alpha^T(t_2,t_1) = u_\alpha((\underset{\sim}{\partial} + ie\underset{\sim}{A}(x,-t')),A_o(\underset{\sim}{x},-t');\ t' \in (-t_1,-t_2)).$$

$$(4.3)$$

The relations (4.1) become:

$$u_\alpha((\underset{\sim}{\partial} - ie\underset{\sim}{A}(x)),\ A_o(x);\ t \in (t_1,t_2)) \qquad (4.4a)$$

$$= \varepsilon_\alpha^U u_\alpha^*((\underset{\sim}{\partial} - ie\underset{\sim}{A}(x)),\ A_o(x);\ t \in (t_2,t_1)) \qquad (4.4b)$$

$$= \varepsilon_\alpha^C u_\alpha^*(-(\underset{\sim}{\partial} - ie\underset{\sim}{A}(x)),-A_o(x);\ t \in (t_1,t_2)) \qquad (4.4c)$$

$$= \varepsilon_\alpha^P u_\alpha(-(\underset{\sim}{\partial} - ie\underset{\sim}{A}(x)),\ A_o(x);\ t \in (t_1,t_2)) \qquad (4.4d)$$

$$= \varepsilon_\alpha^T u_\alpha^*((\underset{\sim}{\partial} - ie\underset{\sim}{A}(\underset{\sim}{x},-t')),A_o(\underset{\sim}{x},-t');t' \in (-t_1,-t_2)). \qquad (4.4e)$$

The relations (4.4) give some information about the functional dependence of the u_α on $\underset{\sim}{\partial}$, A and the time "integration". A comparison of (4.4c) and (4.4d) shows the effect of replacing A_o by $-A_o$. The same expressions determine which u_α are real and which are imaginary when $A_o \equiv 0$. Examining (4.4a) and (4.4d), some aspects of the dependence on $(\underset{\sim}{\partial} - ie\underset{\sim}{A})$ are investigated. The time "integration" may be studied using (4.4a), (4.4b) and (4.4e).

While all these observations of the functional dependence of the u_α have been made with the hope of actually calculating them, nothing more than a vague feeling for their structure has emerged.

5) <u>Fourier Transforms</u>

The Fourier transforms of u and the u_α are needed in (III-6.12). Using

$$\tilde{u}'(\underset{\sim}{p};\underset{\sim}{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x' e^{+i\underset{\sim}{p}.\underset{\sim}{x}'} u'(\underset{\sim}{x}') e^{-i\underset{\sim}{k}.\underset{\sim}{x}'} , \qquad (5.1)$$

the results of sections 3) and 4) may be reexpressed. The relations for $\tilde{u}$ are:

$$\begin{aligned} \tilde{u}(\underset{\sim}{p},\underset{\sim}{k};t_2,t_1) &= [\tilde{u}(\underset{\sim}{k},\underset{\sim}{p};t_1,t_2)]^\dagger \\ &= -\gamma_2[\tilde{u}^C(-\underset{\sim}{p},-\underset{\sim}{k};t_2,t_1)]^*\gamma_2 \\ &= \gamma_o[\tilde{u}^P(-\underset{\sim}{p},-\underset{\sim}{k};t_2,t_1)]\gamma_o \\ &= \gamma_3\gamma_1[\tilde{u}^T(-\underset{\sim}{p},-\underset{\sim}{k};-t_2,-t_1)]^*\gamma_1\gamma_3. \qquad (5.2) \end{aligned}$$

For the $\tilde{u}_\alpha$ it is seen that:

$$\tilde{u}_\alpha(\underset{\sim}{p},\underset{\sim}{k};t_2,t_1) = \varepsilon_\alpha^U[\tilde{u}_\alpha(\underset{\sim}{k},\underset{\sim}{p};t_1,t_2)]^\dagger$$

$$= \varepsilon_\alpha^C[\tilde{u}_\alpha^C(-\underset{\sim}{p},-\underset{\sim}{k};t_2,t_1)]^*$$

$$= \varepsilon_\alpha^P[\tilde{u}_\alpha^P(-\underset{\sim}{p},-\underset{\sim}{k};t_2,t_1)]$$

$$= \varepsilon_\alpha^T(\tilde{u}_\alpha^T(-\underset{\sim}{p},-\underset{\sim}{k};-t_2,-t_1)]^* \ . \qquad (5.3)$$

Both $\tilde{u}$ and the $\tilde{u}_\alpha$ have $\underset{\sim}{\partial}$ dependence which has not been explicitly indicated.

It is difficult to interpret the relations (5.2) and (5.3) in a manner similar to that used in 4). While the u_α are built out of combinations of A_μ, the $\tilde{u}_\alpha$ are not built out of combinations of $\tilde{A}_\mu$. This is because the Fourier transform of a product is not the product of the Fourier transforms.

It is also for this reason that useful relations have not been obtained by taking the Fourier transform of the differential equations (2.2) and the composition law (1.4).

Since more information is available about u and the u_α than about $\tilde{u}$ and the $\tilde{u}_\alpha$, studying (III-6.14) promises more success than studying (III-6.12) in the calculation of pair creation amplitudes.

CHAPTER VI

CALCULATION OF PAIR CREATION AMPLITUDES

1) Introduction

The problem of pair creation in a weak external field is re-examined. From (III-6.14) it is known that:

$$M_1 M_3^*(\underset{\sim}{p},s,\underset{\sim}{q},r) = \frac{m\sqrt{(\omega(p)+m)(\omega(q)+m)}}{2(2\pi)^6\ \omega^{\frac{1}{2}}(p)\,\omega^{\frac{1}{2}}(q)} w^s(0)^\dagger \left[(I + \frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}}{\omega(p)+m}) \times \right.$$

$$\int d^3x\ d^3y\ e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} u(\underset{\sim}{x})(fI + f_o\gamma_o + \underset{\sim}{f}_o\cdot\gamma_o\underset{\sim}{\gamma})u^\dagger(y)e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}$$

$$\left. (I + \frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{q}}{\omega(q)+m}) \right] v^r(0) \ , \tag{1.1}$$

where

$$f = \frac{(2\pi)^3}{2m}\ \delta(\underset{\sim}{x}-\underset{\sim}{y}) \ ,$$

$$f_o = \frac{-2\pi m}{|\underset{\sim}{x}-\underset{\sim}{y}|}\ K_1(m|\underset{\sim}{x}-\underset{\sim}{y}|) \ ,$$

and

$$\underset{\sim}{f}_o = \frac{-2\pi i m}{|\underset{\sim}{x}-\underset{\sim}{y}|^2}\ (\underset{\sim}{x}-\underset{\sim}{y})K_2(m|\underset{\sim}{x}-\underset{\sim}{y}|) \ .$$

If $M_1M_3^*$ is identically zero, all pair creation amplitudes χ are also zero and no creation occurs.

On physical grounds, (1.1) is expected to vanish if the external field is very weak: a pair can't be created unless an energy of 2m is in some sense available. The considerations of Chapter V may be useful in further

examining (1.1) but, unfortunately, it will be seen that still not enough is known about the time translation operator u to prove or even state concisely this physical premonition.

2) Detailed Computation of $M_1 M_3^*$

The vectors $w^s(0)$ and $v^r(0)$ appear in (III-6.7). From their structure, $M_1 M_3^*$ is seen to vanish identically if and only if the upper right hand 2×2 block matrix of the following matrix is zero:

$$M = \int d^3x\, d^3y \left(I + \frac{\underset{\sim}{\gamma} \cdot \underset{\sim}{p}}{\omega(p)+m}\right) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} u(\underset{\sim}{x}) (fI + f_o\gamma_o + \underset{\sim}{f}_o \cdot \gamma_o \underset{\sim}{\gamma}) u^\dagger(\underset{\sim}{y}) e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}$$

$$\left(I + \frac{\underset{\sim}{\gamma} \cdot \underset{\sim}{q}}{\omega(q)+m}\right) . \tag{2.1}$$

If the matrix M is decomposed using the basis of Dirac matrices,

$$M = \sum_\alpha M_\alpha \Gamma_\alpha \quad ,$$

then $M_1 M_3^* \equiv 0$ if and only if

$$\underset{\sim}{M} + \underset{\sim}{M}_o = 0$$

and

$$M_{o5} + M_5 = 0 \; . \tag{2.2}$$

In other words, the coefficients of $\underset{\sim}{\gamma}$, $\gamma_o \underset{\sim}{\gamma}$, $\gamma_o \gamma_5$ and γ_5 in (2.1) are the quantities of interest in finding $M_1 M_3^*$.

The matrix structure of u(x) and u (y) is written:

$$u(\underset{\sim}{x}) = \sum_\alpha u_\alpha((\partial_{\underset{\sim}{x}} - ie\underset{\sim}{A}(x), A_o(x))\Gamma_\alpha = \sum_\alpha u_\alpha((-\overleftarrow{\partial}_{\underset{\sim}{x}} - ie\underset{\sim}{A}(x), A_o(x))\Gamma_\alpha,$$

and

$$u^\dagger(\underset{\sim}{y}) = \sum_\alpha \varepsilon^u_\alpha u^*_\alpha(\ (-\partial_{\underset{\sim}{y}} + ie\underset{\sim}{A}(y), A_o(y))\Gamma_\alpha = \sum_\alpha \varepsilon^u_\alpha u^*_\alpha(-(\partial_{\underset{\sim}{y}} - ie\underset{\sim}{A}(y)), A_o(y))\Gamma_\alpha \ . \qquad (2.3)$$

In carrying out the matrix multiplication of (2.1) step by step, the following abbreviations are used:

$$\sum_\alpha U_\alpha \Gamma_\alpha = (I + \frac{\underset{\sim}{\gamma} \cdot \underset{\sim}{p}}{\omega(p)+m})\ u(\underset{\sim}{x}) \ ,$$

$$\sum_\beta V_\beta \Gamma_\beta = u^\dagger(\underset{\sim}{y})\ \ (I + \frac{\underset{\sim}{\gamma} \cdot \underset{\sim}{q}}{\omega(q)+m}) \ ,$$

and

$$\sum_\delta W_\delta \Gamma_\delta = \sum_\alpha \sum_\beta\ U_\alpha(\underset{\sim}{x})\Gamma_\alpha(fI + f_o\gamma_o + \underset{\sim}{f}_o \cdot \gamma_o\underset{\sim}{\gamma})V_\beta(\underset{\sim}{y})\Gamma_\beta \ . \qquad (2.4)$$

With these definitions,

$$M_\alpha = \int d^3x\ d^3y\ e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\ W_\alpha(\underset{\sim}{x},\underset{\sim}{y})e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}} \ . \qquad (2.5)$$

The coefficients of $\underset{\sim}{\gamma}$, $\gamma_o\underset{\sim}{\gamma}$, γ_{o5} and γ_5 in (2.4) are sought:

$$\begin{aligned}
\underset{\sim}{W} + \underset{\sim}{W}_o = {} & (U_I + U_o)[(f+f_o)(\underset{\sim}{V} + \underset{\sim}{V}_o) + \underset{\sim}{f}_o(V_I - V_o) + \underset{\sim}{f}_o \times (\underset{\sim}{V}_\sigma - \underset{\sim}{V}_{o\sigma})] \\
& + (\underset{\sim}{U} + \underset{\sim}{U}_o)[(f-f_o)(V_I - V_o) + \underset{\sim}{f}_o \cdot (\underset{\sim}{V} + \underset{\sim}{V}_o)] \\
& + (\underset{\sim}{U} + \underset{\sim}{U}_o) \times [(f-f_o)(\underset{\sim}{V}_\sigma - \underset{\sim}{V}_{o\sigma}) + \underset{\sim}{f}_o(V_{o5} + V_5) - \underset{\sim}{f}_o \times (\underset{\sim}{V} + \underset{\sim}{V}_o)] \\
& + (U_{o5} + U_5)[-(f-f_o)(\underset{\sim}{V}_\sigma - \underset{\sim}{V}_{o\sigma}) - \underset{\sim}{f}_o(V_{o5} + V_5) + \underset{\sim}{f}_o \times (\underset{\sim}{V} + \underset{\sim}{V}_o)]
\end{aligned}$$

$$+(\underset{\sim}{U}_\sigma+\underset{\sim}{U}_{o\sigma})[-(f+f_o)(V_{o5}+V_5)-\underset{\sim}{f}_o\cdot(\underset{\sim}{V}_\sigma-\underset{\sim}{V}_{o\sigma}\ ]$$

$$+(\underset{\sim}{U}_\sigma+\underset{\sim}{U}_{o\sigma})\times[(f+f_o)(\underset{\sim}{V}+\underset{\sim}{V}_o)+\underset{\sim}{f}_o(V_I-V_o)+\underset{\sim}{f}_o\times(\underset{\sim}{V}_\sigma-\underset{\sim}{V}_{o\sigma})] \qquad (2.6)$$

and

$$W_{o5}+W_5=(U_I+U_o)[(f+f_o)(V_{o5}+V_5)+\underset{\sim}{f}_o\cdot(\underset{\sim}{V}_\sigma-\underset{\sim}{V}_{o\sigma})]$$

$$+(\underset{\sim}{U}+\underset{\sim}{U}_o)\cdot[(f-f_o)(\underset{\sim}{V}_\sigma-\underset{\sim}{V}_{o\sigma})+\underset{\sim}{f}_o(V_{o5}+V_5)-\underset{\sim}{f}_o\times(\underset{\sim}{V}+\underset{\sim}{V}_o)]$$

$$+(U_{o5}+U_5)[(f-f_o)(V_I-V_o)+\underset{\sim}{f}_o\cdot(\underset{\sim}{V}+\underset{\sim}{V}_o)]$$

$$+(\underset{\sim}{U}_\sigma+\underset{\sim}{U}_{o\sigma})\cdot[(f+f_o)(\underset{\sim}{V}+\underset{\sim}{V}_o)+\underset{\sim}{f}_o(V_I-V_o)+\underset{\sim}{f}_o\times(\underset{\sim}{V}_\sigma-\underset{\sim}{V}_{o\sigma})] \qquad (2.7)$$

From (2.2), $M_1M_3^* \equiv 0$ if and only if

$$\int d^3x\, d^3y\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}(\underset{\sim}{W}+\underset{\sim}{W}_o)e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}} = 0\ ,\ \forall\ \underset{\sim}{p},\underset{\sim}{q}$$

and
$$\int d^3x\, d^3y\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}(W_{o5}+W_5)e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}=0,\ \forall\ \underset{\sim}{p},\underset{\sim}{q}\ . \qquad (2.8)$$

Use of the definition (2.4) for U_α and V_β in (2.6) and (2.7) proves to be of little computational value - no cancellations occur. A careful examination of such an expansion of (2.6) and (2.7) does, however, demonstrate that every combination of the u_α, f_β, and u_γ^* appears and, with appropriate factors of $p_i/\omega(p)+m$, $q_j/\omega(q)+m$, $e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}$ and $e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}$, must be integrated.

At this point an impasse has been reached. The symmetry identities of Chapter V tell little about a quantity like:

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, u_\alpha((-\overleftarrow{\partial}_{\underset{\sim}{x}}-ie\underset{\sim}{A}(x),A_o(x))f_\beta(\underset{\sim}{x}-\underset{\sim}{y})u^*_\gamma(-(\partial_{\underset{\sim}{y}}-ie\underset{\sim}{A}(\underset{\sim}{y})),A_o(\underset{\sim}{y}))e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}.$$

Even in the simplifying case of $\underset{\sim}{A} \equiv 0$ and $\underset{\sim}{p} = \underset{\sim}{q} = 0$, not enough is known of u_α's dependence on A_o to accomplish the necessary integrations.

3) <u>Conclusions</u>

The c-number time translation operator u is the quantity upon which all time development, including pair creation, depends. It is possible to know only as much about pair creation as is known about u.

A first order calculation of u permitted examination of pair creation to first order. The work of Chapter IV showed that in pair creation, to this order, a "conservation of momentum and energy" relation holds. The higher order calculations indicated that momentum was "conserved" at each vertex in a proper Feynman diagram.

The exact calculation of the pair creation amplitude through the programme of Bogoliubov transformations as outlined in Chapters II and III and as further attempted in this chapter, has proved impossible as a result of a scarcity of information about the time translation operator u.

Before any threshold effect in pair creation may be established using a non-perturbative approach, the

c-number time translation operator must be better known. It is only after more details of the functional structure of u have been tabulated that questions, such as the above, which are related to time development in the external field problem may be answered.

CHAPTER VII

THE ADIABATIC THEOREM

1) The Adiabatic Theorem

A useful computational technique in quantum mechanics and quantum field theory is the adiabatic switching of interactions. When it is difficult to calculate an integral with potential $A(\underset{\sim}{x},t)$, it is often easier to calculate it with $A'(\underset{\sim}{x},t) = e^{-\alpha|t|} A(\underset{\sim}{x},t)$, where α is some small positive real number, and then to take the limit $\alpha \to 0$. This manipulation is usually dismissed by saying that at large times the particles of the system are far from each other or far away from the region of strong potential. At such times little interaction occurs and it may as well be turned off. While such an argument may make sense on physical grounds, careful mathematical examination of the limiting procedure is required before adiabatic switching may be legitimately employed.

The adiabatic theorem was proved for c-number quantum mechanics many years ago. Born and Fock [1928] state the theorem this way:

"Label the states of a system with the quantum numbers of the corresponding energy levels. The adiabatic theorem then asserts that if the system is initially in a state with a definite quantum

number an adiabatic change in potential results in only infinitesimal transition probabilities into states of other quantum numbers, even though the energy levels may themselves be shifted by finite amounts."

While the energy levels of an adiabatically switched problem may be different from those of the original problem, switching does not change the population of states. That this has been proved valid for quantum mechanics justifies the use of adiabatic switching for problems in this subject.

In quantum field theory, the presence of infinitely many degrees of freedom has made the proof of the adiabatic theorem difficult. At present, it remains a conjecture. Ironically, in quantum field theory adiabatic switching is an essential tool in a discipline where many other things are already difficult.

Källén [1972] emphasizes the importance of the adiabatic theorem in the fully interacting theories for the interpretation of particles. Particles must be defined using free fields - this definition may be made in terms of any available set of free fields. One set of physical interest consists of the auxiliary fields $A_\mu^{(o)}(x;T)$ and $\psi^{(o)}(x;T)$ specially constructed for each time T. The incoming free fields $A_\mu^{(o)}(x)$ and $\psi^{(o)}(x)$ are also important. The eigenstates of $H^{(o)}(T) =$

$H^{(o)}(A_\mu^{(o)}(x;T),\ \psi^{(o)}(x;T))$ define the particles that the system would contain if the interaction remained constant after time T. The eigenstates of $H^{(o)} = H^{(o)}(A_\mu^{(o)}(x),\ \psi^{(o)}(x))$ define particles which are experimentally more meaningful - these are the incoming particles which have existed in the system for a long time. To have a useful particle interpretation in an interacting theory it is necessary to turn the interaction on and off slowly during the long existence of the particles. Without an adiabatic theory to guarantee that this switching operation doesn't change the creation and scattering amplitudes of interest, a fully interacting theory has no interpretation in terms of particles.

In the external field problem, the adiabatic theorem plays a similar role. Free particles and a potential $e^{-\alpha|t|}A(\underset{\sim}{x},t)$ are expected to yield a description of the physics. Another method, using particles as defined through auxiliary fields and the actual potential $A(\underset{\sim}{x},t)$, has been developed in Chapters II and III to give the physics as well. Any proof of the adiabatic theorem will have to show the equivalence of these descriptions.

2) An Example

The following example in c-number quantum mechanics illustrates the use of the adiabatic theorem. Since a close analogy may be drawn between this situation and the external field problem, the details of this calculation will give insight into the physical principles anticipated for an external field problem.

Consider a time dependent simple harmonic oscillator with Hamiltonian:

$$H(t) = \frac{p^2}{2m} + \frac{1}{2} m^2\omega^2(t)x^2 = \hbar\omega(t)[a^\dagger(t)a(t) + \tfrac{1}{2}] \quad . \qquad (2.1)$$

The state of n quanta at time t is

$$\Psi_n^t(x) = (a^\dagger(t))^n|0\rangle_t \qquad (2.2)$$

and its energy is

$$E_n(t) = (n + \tfrac{1}{2})\hbar\omega(t) \quad . \qquad (2.3)$$

The time dependence of the frequency $\omega(t)$ is chosen as:

$$\begin{aligned} \omega(t) &= \omega && t < 0 \\ &= \omega + \omega' e^{-\alpha t} && t \geq 0 \ , \quad \alpha << 1 \ . \end{aligned} \qquad (2.4)$$

At t = 0 the frequency suddenly jumps and then is slowly switched off at large times. At each time t, the wave function may be expressed in terms of the complete set $\{\Psi_n^t\}$:

$$\Psi(t,x) = \sum a_n \Psi_n^{in}(x) e^{-iE_n^{in} t/\hbar} \qquad t < 0 ,$$

$$= \sum b_n(0) \Psi_n^{o}(x) e^{-iE_n^{o} t/\hbar} \qquad t = 0 ,$$

$$= \sum b_n(t) \Psi_n^{t}(x) e^{-i\int_0^t E_n(\tau) d\tau/\hbar} \qquad t > 0 . \qquad (2.5)$$

The sudden change of frequency at t = 0 disturbs the population of the levels so it is expected that $a_n \neq b_n(0)$. The adiabatic theorem states that slow change in interaction results in no transitions, so $b_n(0) = b_n(t)$ should hold in the limit of $\alpha \to 0$. Notice that the energy of the nth level drops by $n\hbar\omega'$ while the frequency switches from $(\omega + \omega')$ back to ω.

Suppose the system is in the vacuum state (n = 0) for t < 0:

$$\Psi(t,x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar} x^2\right) e^{-i\frac{\omega}{2}t} , \quad t < 0 . \qquad (2.6)$$

The wavefunction is continuous in time so that:

$$\Psi(0,x) = \left(\frac{m\omega}{\hbar\pi}\right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right) = \sum b_n(0) \Psi_n^{o}(x) ,$$

with

$$b_n(0) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \left(\frac{m(\omega+\omega')}{\hbar\pi}\right)^{\frac{1}{4}} \left(\frac{1}{2^n n!}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{m(2\omega+\omega')}{2\hbar} x^2\right) \times$$

$$H_n\left(\sqrt{\frac{m(\omega+\omega')}{\hbar}}\, x\right) . \qquad (2.7)$$

The coefficient $b_n(0)$ gives the probability amplitude of

the system being in the state n at the time t = 0, given that it was in the vacuum state before the frequency jumped.

Using the adiabatic approximation as outlined by A.Z. Capri [1972], the amplitude for the system to leave the state n for the state k before a later time t is:

$$d_k^n(t) = \frac{1}{i\hbar(\omega_{kn}^o)^2} \left(\Psi_k^o(x),\ \frac{\partial H}{\partial t}\bigg|_0 \Psi_n^o(x)\right)\left(e^{i\omega_{kn}^o t} - 1\right) \qquad k \neq n \tag{2.8}$$

where $\omega_{kn}^o = (k-n)\omega(0) = (k-n)(\omega+\omega')$.

Since

$$\frac{\partial H}{\partial t}\bigg|_{t=0} = m^2(\omega+\omega')(-\alpha\omega')(x^2)$$

the calculation of (2.8) may proceed:

$$d_k^n(t) = -\frac{\alpha}{i(k-n)^2}\frac{m\omega'}{(\omega+\omega')^2}\left(\frac{1}{2}((n)(n+1))^{\frac{1}{2}}\delta_{k,n+2} + \frac{1}{2}((n)(n-1))^{\frac{1}{2}}\delta_{k,n-2}\right)\left(e^{i(k-n)(\omega+\omega')t} - 1\right). \tag{2.9}$$

Using the definitions of $b_n(t)$ and $d_k^n(t)$:

$$b_k(t) = b_k(0) + \sum_{n\neq k} d_k^n(t) = b_k(0) + o(\alpha)\ . \tag{2.10}$$

Only infinitesimally few transitions occur for t > 0. From (2.9), $d_k^n(t)$ is a bounded function of time t for all times, so for all times t:

$$\lim_{\alpha \to 0} b_k(t) = b_k(0) \; . \tag{2.11}$$

This is the result expected from the adiabatic theorem.

A system of n noninteracting bosons of energy $\omega(t)$ may be compared to a time dependent simple harmonic oscillator in state Ψ_n^t. Considering the external field problem for bosons is, therefore, similar to working the above example. Suppose the external field were kept constant for $t < 0$, jumped suddenly at $t = 0$ and then were slowly switched off for large times. The relations (2.7), (2.10) and (2.11) suggest that if the system were initially in the vacuum state that many particles would be created at $t = 0$ and that these particles would remain in the system, with eventually lower energies, at large times. The creation of many particles by a jump in the external field has been demonstrated by Labonté [1973]. A proof of the adiabatic theorem for this particular potential is needed to verify the suggested large time behaviour.

3) The External Field Problem

For an external field A, particles may be defined at each time t_o using the auxiliary time independent field $A(\underset{\sim}{x}, t_o)$. The vacuum and particle states lie in a Fock Hilbert space $\mathcal{H}_{t_o}$. When the Bogoliubov transformations relating the creation/annihilation operators

at one time t_1 to those at another time t_2 are unitarily implementable, the corresponding Fock Hilbert spaces $\mathcal{H}_{t_1}$ and $\mathcal{H}_{t_2}$ are identical.

The adiabatically switched problem with $A'(\underset{\sim}{x},t) = e^{-\alpha|t|}A(\underset{\sim}{x},t)$ may also be treated with such a Bogoliubov transformation programme. At each time t_o there is a Fock Hilbert space $\mathcal{H}^{\alpha}_{t_o}$ of particle states. Since $A'(\underset{\sim}{x},0) = A(\underset{\sim}{x},0)$, $\mathcal{H}_o = \mathcal{H}^{\alpha}_o$. If the Bogoliubov transformations for A' are unitarily implementable, $\mathcal{H}^{\alpha}_{t_1} = \mathcal{H}^{\alpha}_{t_2}$ for all t_1 and t_2.

In the case that both $A(\underset{\sim}{x},t)$ and $e^{-\alpha|t|}A(\underset{\sim}{x},t)$ give rise to unitarily implementable Bogoliubov transformations, all the Fock Hilbert spaces defined above are identical. Since the adiabatically switched potential vanishes at large times, the Fock Hilbert space appropriate for calculations with $A(\underset{\sim}{x},t)$ or $e^{-\alpha|t|}A(\underset{\sim}{x},t)$ is the free Fock Hilbert space $\mathcal{H}^{in}$ defined in Chapter II.

One version of the adiabatic theorem for the external field problem is that scattering and creation amplitudes for a very slowly switched problem may differ only infinitesimally from those of the unswitched problem.

Time Independent External Field

From the discussion of Chapter II, it is known that no creation occurs in a time independent external field. If the problem with potential $A(\underset{\sim}{x})$ were treated

using switching, time dependence would be present and the possibility of pair creation would arise. For the adiabatic theorem to hold, it is necessary that the pair creation amplitude χ_α for the potential $e^{-\alpha|t|}A(\underset{\sim}{x})$ be infinitesimal for very small α.

G. Labonté [1974] has recently been studying the circumstances under which the Bogoliubov transformations for $e^{-\alpha|t|}A(\underset{\sim}{x})$ are unitarily implementable. It is necessary for the Bogoliubov transformations to be unitarily implementable to have a physical particle description of the system at all times.

Using the notation of Chapter III, the criterion for unitary implementability is:

$$\int d\gamma d\beta |M_2^{(\alpha)}(\gamma,\beta)|^2 < \infty \quad ,$$

$$\int d\gamma d\beta |M_3^{(\alpha)}(\gamma,\beta)|^2 < \infty \quad . \tag{3.1}$$

It is sufficient to examine this relation for a Bogoliubov transformation relating time t = 0 and a later time t'. If $f_{\varepsilon\beta}^{t=0}(\underset{\sim}{x})$ are the c-number solutions to the Dirac equation using $A(\underset{\sim}{x})$ and $f_{\varepsilon\beta}^{t'}(\underset{\sim}{x})$ are the c-number solutions to the Dirac equation using $e^{-\alpha t'}A(\underset{\sim}{x})$,

$$M_2^{(\alpha)}(\gamma,\beta) = (f_{+\gamma}^{t'}, u_\alpha(t',0)f_{-\beta}^{t=0}) \quad ,$$

$$M_3^{(\alpha)}(\gamma,\beta) = (f_{-\gamma}^{t'}, u(t',0)f_{+\beta}^{t=0}) \quad , \tag{3.2}$$

where

$$i \frac{\partial}{\partial t'} u(t',t) = (h_o - e\gamma^o \gamma \cdot e^{-\alpha|t'|} A(\underset{\sim}{x}))u(t',t). \qquad (3.3)$$

G Labonté carried out the integrations involved and found that the switched problem does have a physical particle description at all times if the matrix $-e\gamma^o\gamma \cdot A(\underset{\sim}{x})$ is a bounded, square integrable function of $\underset{\sim}{x}$. That is

1) $\exists\ a < \infty \ni (\phi, [-e\gamma^o\gamma \cdot A]^2\phi) \leq a^2(\phi,\phi) \quad \forall\ \phi \in \mathcal{H}$, and

2) $\int d^3x \ \text{Trace}\ (-e\gamma^o\gamma \cdot A(\underset{\sim}{x}))^2 < \infty$.

These two conditions are equivalent and reduce to:

$$\int d^3x \ (A_o^2(\underset{\sim}{x}) + A(\underset{\sim}{x}) \cdot A(\underset{\sim}{x})) < \infty \ .$$

That the Bogoliubov transformations are unitarily implementable also means:

$$|0(t=0)\rangle = T^{(\alpha)}(t',0)\,|0(t')\rangle^{(\alpha)} \qquad (3.4)$$

where $T^{(\alpha)}(t',0)$ is a unitary q-number operator. The vacuum state $|0(t')\rangle^{(\alpha)}$ is the physical vacuum at the time t' for the problem with potential $e^{-\alpha|t|}A(\underset{\sim}{x})$. Another statement of the adiabatic theorem would claim that (in a strong operator sense):

$$T^{(\alpha)}(t',0) = I + o(\alpha) \ . \qquad (3.5)$$

If the system were in the vacuum state at time $t = 0$,

the adiabatically switched external field could cause transitions to states other than $|0(t')>^{(\alpha)}$ with a probability amplitude of at most $o(\alpha)$. So, for very small α, pair creation and other creation amplitudes would be infinitesimal.

Perturbation Approach

In Chapter IV, perturbation expansions were found for the time translation operator, the Bogoliubov matrices and the pair creation amplitude. It may be possible to write converging expansions for these quantities for both switched and unswitched problems. The adiabatic theorem would demand that the calculations for switched and unswitched potentials differ only infinitesimally in the limit of very slow switching.

The first order contribution to pair creation is given by (IV-3.11):

$$\chi^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi}\frac{m}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \gamma^o \gamma\cdot\tilde{\tilde{A}}(-\underset{\sim}{p}-\underset{\sim}{q},\omega(p)+\omega(q))v^r(\underset{\sim}{q}) \tag{3.6}$$

The first order contribution to pair creation for the switched problem is:

$$\chi_\alpha^{(1)}(\underset{\sim}{p}s,\underset{\sim}{q}r) = \frac{1}{2\pi}\frac{m}{\sqrt{\omega(p)\omega(q)}} w^s(\underset{\sim}{p})^\dagger \gamma^o \gamma\cdot\tilde{\tilde{A}}'(-\underset{\sim}{p}-\underset{\sim}{q},\omega(p)+\omega(q))v^r(\underset{\sim}{q}) \tag{3.7}$$

where $A'(\underset{\sim}{x},t) = e^{-\alpha|t|}A(\underset{\sim}{x},t)$,

and $$\tilde{\tilde{A}}'(\underset{\sim}{p},\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\alpha}{\alpha^2+(\omega-\omega')^2} \tilde{\tilde{A}}(\underset{\sim}{p},\omega') \ . \tag{3.8}$$

So $\tilde{\tilde{A}}$ is averaged over a region of width of order α to obtain $\tilde{\tilde{A}}'$.

Consider a potential A whose time Fourier transform is bounded and has no components for $|\omega| \geq 2m$:

$$|\tilde{\tilde{A}}(\omega)| \leq \theta(2m-\omega)\theta(2m+\omega)A_o \ . \tag{3.9}$$

From (3.8), for $|\omega| > 2m$:

$$|\tilde{\tilde{A}}'(\omega)| \leq \frac{1}{\pi} \int_{-2m}^{2m} d\omega' \frac{\alpha}{\alpha^2+(\omega-\omega')^2} A_o \leq \frac{4m\ A_o}{\pi(|\omega|-2m)^2}\ \alpha \ . \tag{3.10}$$

The unswitched potential does not have sufficiently large Fourier components to cause creation (by (3.6)) and the switched potential may (by (3.7)) only give a pair creation amplitude of order α.

That the adiabatic theorem holds true term by term in the perturbation expansions is clear from the integrations of Chapter IV, sections 2) and 3). All the doubt of the validity of the adiabatic theorem is around the question of convergence. If the expansions for the unswitched problem converge, will those of an adiabatically switched problem? An infinite series of continuous functions is not always continuous - is it possible that while term by term adiabatic switching

works, it doesn't for the complete calculation? What constraints are needed on the spatial and time behaviour of a potential for convergence of the different perturbation series? Further understanding of the nature of convergence of the perturbation expansions could lead to resolution of the adiabatic theorem quandary.

Footnotes

1) Meaning of Lorentz Invariance in External Field Problem

Even in the presence of an external field, the theory has a certain kind of Lorentz invariance. The transformation properties under a Lorentz transformation of all the quantities in the theory are known: ψ is a spinor, A is a vector, etc. Knowing the external field in one Lorentz frame allows one to calculate the external field in any other Lorentz frame. An observer in a new Lorentz frame may ask the same sorts of questions about creation and scattering of electrons and positrons interacting with his external field as the observer did in the old Lorentz frame. The new observer's theory is exactly like that of the old observer except that all old quantities appearing in the theory have been replaced by new (transformed) ones. In the sense that the theory takes the same form with identically appearing equations in all Lorentz frames, the theory is Lorentz invariant.

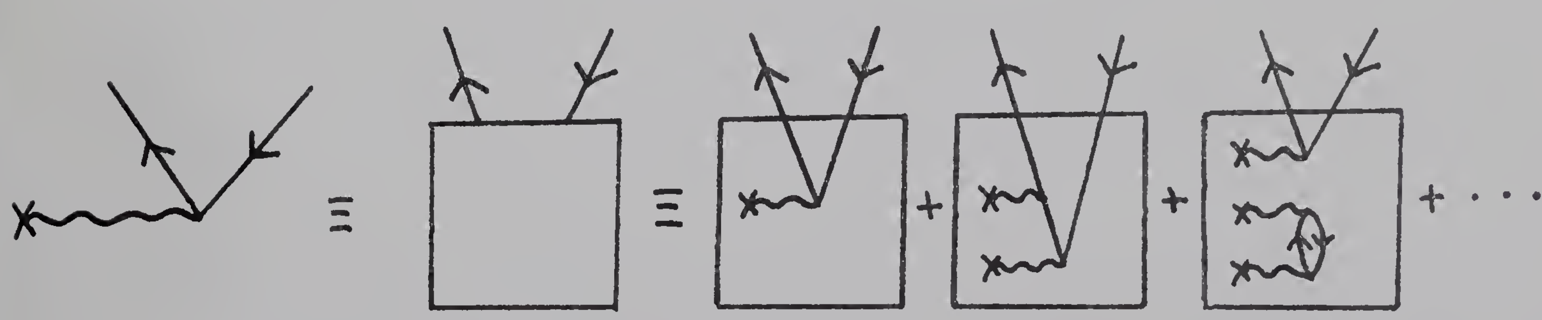
2) "Feynman" Diagrams

The diagrams of Fig. 1 are not proper Feynman diagrams - they show physical particles, not the "in" particles usually drawn. These diagrams illustrate the amplitude of processes taking place between times t_1 and t_2 - the particles at the bottom of a diagram are particles as defined at time t_1; those at the top of a

diagram are particles as defined at time t_2. In the vertex is buried all possible intermediate states that may have occurred between times t_1 and t_2.

In Fig. 2, the vertex for pair creation is drawn as a box and some of its contents illustrated. Again, these contents have been drawn as physical particles. Notice that there are contributions of all orders in e to $\chi(\alpha,\beta)$ - this may also be seen from equation (IV-3.3). By examining physical particles the renormalization difficulties of using "in" particles have been avoided - the scattering and creation amplitudes are finite even though infinitely many processes contribute.

Fig. 2: Contributions to the Pair Creation Amplitude



To draw a proper Feynman diagram for one of the diagrams in Fig. 1, each physical particle would need to be replaced by an appropriate superposition of "in" particles, pairs, etc. and the contents of the vertex also expanded. In the special case when the external field is zero except for times between t_1 and t_2, the physical particles at times t_1 and t_2 coincide with "in" particles and only the vertex needs re-drawing. The final appearance of the Feynman diagram would be similar to Fig. 2 but have different meaning. The terms in the perturbation expansion of Chapter IV may be illustrated directly by true Feynman diagrams.

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